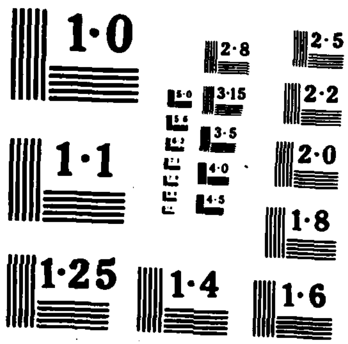
AD-A170 187 UPPER SEMICONTINUITY OF ATTRACTORS FOR APPROXIMATIONS OF SEMIGROUPS AND P.. (U) BROWN UNIV PROVIDENCE RI LEFSCHETZ CENTER FOR DYNAMICAL SYSTE.. J K HALE ET AL. 1/1

UNCLASSIFIED OCT 85 LCDS-85-29 AFOSR-TR-86-0374 F/G 12/1 NL


1·0
2·8
2·5
3·15
2·2
3·5
1·1
2·0
4·0
4·5
1·8
1·25
1·4
1·6

AFOSR-TR- 86-0374

2

UPPER SEMICONTINUITY OF ATTRACTORS FOR APPROXIMATIONS OF SEMIGROUPS AND PARTIAL DIFFERENTIAL EQUATIONS

by

Jack K. Hale, Xiao-Biao Lin and Geneviève Raugel

October 1985 LCDS #85-29

AD-A170 187

Lefschetz Center for Dynamical Systems

Division of Applied Mathematics Brown University Providence RI 02912

UPPER SEMICONTINUITY OF ATTRACTORS FOR APPROXIMATIONS OF SEMIGROUPS AND PARTIAL DIFFERENTIAL EQUATIONS

by

Jack K. Hale, Xiao-Biao Lin and Geneviève Raugel

October 1985 LCDS #85-29

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. AD-A170 187	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Upper Semicontinuity of Attractors for Approximations of Semigroups and Partial Differential Equations		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER AFOSR-TR- 86-0374
7. AUTHOR(s) J.K. Hale, X.-B. Lin, and G. Raugel		8. CONTRACT OR GRANT NUMBER(s) AFOSR 84-0376
9. PERFORMING ORGANIZATION NAME AND ADDRESS Lefschetz Center for Dynamical Systems Division of Applied Mathematics Brown University, Providence, RI 02912		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304 A1
11. CONTROLLING OFFICE NAME AND ADDRESS AFOSR/NM Bolling Air Force Base Washington, DC 20332		12. REPORT DATE October 1985
		13. NUMBER OF PAGES 68
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Suppose a given evolutionary equation has a compact attractor and the evolutionary equation is approximated by a finite dimensional system. Conditions are given to ensure the approximate system has a compact attractor which converges to the original one as the approximation is refined. Applications are given to parabolic and hyperbolic partial differential equations.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

86 7 2?

UPPER SEMICONTINUITY OF ATTRACTORS FOR APPROXIMATIONS OF SEMIGROUPS AND PARTIAL DIFFERENTIAL EQUATIONS

by

Jack K. Hale*, Xiao-Biao Lin** and Geneviève Raugel***

*Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, RI 02912

**Department of Mathematics
Michigan State University
East Lansing, MI 48823

*** Ecole Polytechnique
Centre de Mathématiques Appliquées
Unité de Recherche Associée en CNRS-756
91128 Palaiseau Cedex, France

October 1985

This research was supported by the Air Force Office of Scientific Research under Grant #AF-AFOSR 84-0376, by the U.S. Army Research Office under Grant #DAAG-29-83-K-0029, and by the National Science Foundation under Grant #DMS-8507056. UL

TABLE OF CONTENTS

UPPER SEMICONTINUITY OF ATTRACTORS FOR APPROXIMATIONS OF SEMIGROUPS AND PARTIAL DIFFERENTIAL EQUATIONS

by

J. K. Hale, X.-B. Lin and G. Raugel

ABSTRACT

Suppose a given evolutionary equation has a compact attractor and the evolutionary equation is approximated by a finite dimensional system. Conditions are given to ensure the approximate system has a compact attractor which converges to the original one as the approximation is refined. Applications are given to parabolic and hyperbolic partial differential equations.

1. Introduction.

Suppose X is a Banach space and $T(t)$, $t \geq 0$, is a C^r-semigroup on X with $r \geq 0$; that is, $T(t)$, $t \geq 0$, is a semigroup with $T(t)$ continuous in t, x together with the derivatives in x up through the order r.

Following standard terminology (see, for instance, [Hale, 2]), a set $B \subset X$ is said to <u>attract</u> a set $C \subset X$ under the semigroup $T(t)$ if, for any $\epsilon > 0$, ther is a $t_0 = t_0(B,C,\epsilon)$ such that $T(t)C \subset \mathcal{N}(B,\epsilon)$ for $t \geq t_0$, where $\mathcal{N}(B,\epsilon)$ denotes the ϵ-neighborhood of B. A compact invariant set $\mathcal{A}$ is said to be a <u>local attractor</u> if there exits and open neighborhood U of $\mathcal{A}$ such that $\mathcal{A}$ attracts U. The set $\mathcal{A}$ is an <u>attractor</u> if, for any bounded set B in X, $\mathcal{A}$ attracts B. Conditions for the existence of an attractor may be found in [Hale, 2].

Now suppose the semigroup depends on a parameter λ belonging to an open subset of a Banach space, say $T(t) = T_\lambda(t)$, where $T_\lambda(t)x$ is continuous in (t,x,λ), the continuity in λ being uniform on bounded sets. If $\mathcal{A}_{\lambda_0}$ is a local attractor for $T_{\lambda_0}(t)$, *then additional smoothing properties of* $T_\lambda(t)$ *will* imply there is a neighborhood V of λ_0 such that $T_\lambda(t)$, $\lambda \in V$, has a local attractor $\mathcal{A}_\lambda$and $\mathcal{A}_\lambda$ is <u>upper continuous</u> at λ_0, that is, $\delta_X(\mathcal{A}_\lambda,\mathcal{A}_{\lambda_0}) \to 0$ as $\lambda \to \lambda_0$ where, for any two subsets A,B of X,

$$\delta_X(A,B) = \sup_{x\in A} \mathrm{dist}_X(x,B)$$

and $\mathrm{dist}_X(x,B) = \inf_{y\in B} ||x-y||_X$.

The most general result of this type is due to [Cooperman] and may be found also in [Hale, 1]. The result for gradient systems is in [Hale, 2].

The spirit of this paper relates to the above property of upper semicontinuity of a local attractor. Here we consider semigroups $T_h(t)$ depending on a parameter $h > 0$ which "approximate" the semigroup $T(t)$ and give conditions under which there exists a local attractor $\mathcal{A}_h$ for $T_h(t)$ with the property that $\delta(\mathcal{A}_h,\mathcal{A}) \to 0$ as $h \to 0$. The essential difference between the results here and the ones mentioned before is that the approximate semigroups can correspond to Galerkin approximations, splines or discretizations in time of evolutionary equations. These approximations have no uniform continuity property with respect to h.

The outline of the paper is as follows. In Section 2 we give a general approximation result which attempts to bring out the essential elements of the approximate and exact semigroups to ensure that there is a local, compact attractor which is upper semicontinuous. We also give one result in which we assume the approximate semigroups have a local compact attractor and then infer that the exact semigroup has a compact attractor. For the Navier-Stokes equation and the case in which the local attractor for each approximation is a point, Constantin, Foias and Temam have given conditions which ensure that the original equations have an equilibrium. Schmitt, Thompson and Walter discuss the solution of an elliptic boundary value problem in an infinite strip by analyzing solutions of approximate differential equations. This aspect of the problem is important but much more difficult and will be developed further in subsequent publications. The remainder of the paper is devoted to giving specific approximation schemes for particular evolutionary systems for which the hypotheses of Section 2 are satisfied. These applications include spectral projection methods for sectorial evolutionary equations and Galerkin approximations for parabolic equations as well as discretizations in time. Some results about the approximation of the Navier-Stokes equations and of a damped hyperbolic wave equation also are given.

In this paper, the convergence of the attractor $\mathcal{A}_h$ to $\mathcal{A}$ as $h \to 0$ is considered only in the sense of sets. The relationship between the dynamics on the attractors also must be discussed. This problem is much more difficult and requires some knowledge of the flow on A. Some results on the case in which the flow on A is Morse-Smale already have been obtained and will appear in [Lin and Raugel]. For the case of a scalar parabolic equation in one space dimension with a cubic nonlinearity, this latter property has been discussed for space and time approximations using the Conley index [Khalsa]. Numerical computations using Galerkin approximations have been done for a similar example [Rutkowski], [Mora].

2. A General Approximation Result

In this section, we give a general result on the approximation of a local attrractor by "approximate" semigroups. These results are very similar to local versions of the ones of [Cooperman] (see also [Hale, 1]). More precisely, let $h > 0$ be a parameter which will tend to 0 and let $(X_h)_h$ be a family of subspaces of X such that

$$(2.1) \qquad \lim_{h \to 0} \operatorname{dist}_X(x, X_h) = 0, \text{ for any } x \text{ in } X.$$

Let $T_h(t)$, $t \geq 0$, be a C^s-semigroup on X_h with $s \geq 0$. Actually $T_h(t)x_h$ need not be a priori defined for all $t > 0$. More precisely, we shall only assume that $T_h(0) = Id_X$, $T_h(t+s)x_h = T_h(s)T_h(t)x_h$ for $s \geq 0$, $t \geq 0$ (as soon as $T_h(t+s)x_h$ and $T_h(s)T_h(t)x_h$ are well defined), that $T_h(t)x_h$ is continuous in t and x_h when it is defined and finally, that $T_h(t)x_h$ is left-continuous at t_1 if $T_h(t)x_h$ exists on $[t_0,t_1)$. The semigropus $T_h(t)$ are said to <u>conditionally approximate</u> T(t) <u>on a set</u> $U \subset X$ <u>uniformly on an interval</u> $I \equiv [t_0,t_1] \subset R^+$ if there are a constant $h(I,U) > 0$ and a function $\eta(h,I,U)$ defined for $0 < h \leq h(I,U)$ such that

$$(2.2) \qquad \lim_{h \to 0} \eta(h,I,U) = 0$$

and, for any $0 < h \leq h(I,U)$, if $u \in U \cap X_h$ has the property that $T(t)u$, $T_h(t)u$ are defined and belong to U for $t \in [0,t_2]$ where $t_0 < t_2 \leq t_1$, then

$$(2.3) \qquad \|T(t)u - T_h(t)u\|_X \leq \eta(h,I,U) \quad \text{for} \quad t_0 \leq t \leq t_2$$

The semigroups $T_h(t)$ are said to <u>approximate T(t) on a set $U \subset X$ uniformly on an interval $I \subset R^+$</u> if $T_h(t)$ conditionally approximates T(t) on U uniformly on I and if, moreover, for $0 < h \leq h(I,U)$ and any $u \in U \cap X_h$, the functions $T(t)u$, $T_h(t)u$ are defined and satisfy the inequality (2.3) for all $t \in I$.

The semigroups $T_h(t)$ are said to <u>(conditionally) approximate $T(t)$ on $U \subset X$ uniformly on compact sets of R^+</u> if $T_h(t)$ (conditionally) approximates $T(t)$ on U uniformly on any compact interval $I \subset \mathbf{R}^+$. We recall that, in the following, $N(B,\epsilon)$ denotes the ϵ-neighborhood of a set B in the Banach space X.

Lemma 2.1. *Assume that there exist a bounded set* $B_0 \subset X$ *and an open set* $U_0 \supset N(B_0,d_0)$ *for some* $d_0 > 0$ *such that* B_0 *attracts* U_0 *under* $T(t)$. *Moreover, assume that there exist an open set* $U_1 \supset N(B_0,d_1)$ *for some* $d_1 > 0$ *and a constant* $t_0 \geq 0$ *such that* $T_h(t)$ *approximates* $T(t)$ *on* U_1 *uniformly on compact sets of* $[t_0,\infty)$. *Then, for any* $\epsilon_0 > 0$, *there are* $h_0 > 0$ *and* $\tau_0 > t_0$ *such that, for* $0 < h \leq h_0$, *for* $t \geq \tau_0$,

$$T_h(t)(U_0 \cap U_1 \cap X_h) \subset N(B_0,\epsilon_0).$$

Proof. Without any restriction, we can assume that $\epsilon_0 \leq \inf(d_0,d_1)$. As B_0 attracts U_0, there exists $\tau_0 > t_0$ such that, for $t \geq \tau_0$, $T(t)U_0 \subset N(B_0,\epsilon_0/2)$. Thanks to the hypothesis (2.2), there exists $h_0 > 0$ such that, for $h \leq h_0$, $\eta(h,2\tau_0,U_1) \leq \epsilon_0/2$. Therefore, for $h \leq h_0$, for $\tau_0 \leq t \leq 2\tau_0$, $T_h(t)(U_0 \cap U_1 \cap X_h) \subset N(B_0,\epsilon_0)$. Let us remark that $U_0 \cap U_1 \cap X_h \neq \phi$, because $U_0 \cap U_1 \supset N(B_0,\inf(d_0,d_1))$.

Now, let us prove by induction that, for $t \geq \tau_0$, $T_h(t)(U_0 \cap U_1 \cap X_h) \subset N(B_0,\epsilon_0)$. Assume that, for $\tau_0 \leq t \leq h\tau_0$, $T_h(t)(U_0 \cap U_1 \cap X_h) \subset N(B_0,\epsilon_0)$ and let us prove this property for $\tau_0 \leq t \leq (n+1)\tau_0$. If $n\tau_0 \leq t \leq (n+1)\tau_0$, $t = (n-1)\tau_0 + \tau$ with $\tau_0 \leq \tau \leq 2\tau_0$. Let $u_{0h} \in U_0 \cap U_1 \cap X_h$; we have:

$$T_h(t)u_{0h} = T_h(\tau)T_h(n-1)\tau_0)u_{0h}$$

By the induction hypothesis, $T_h((n-1)\tau_0)u_{0h} \in N(B_0,\epsilon_0) \cap X_h$, and hence, $T_h((n-1)\tau_0)u_{0h} \in U_0 \cap U_1 \cup X_h$. Therefore, on the one hand, $T(\tau)T_h((n-1)\tau_0)u_{0h} \in N(B_0,\epsilon_0/2)$, and, on the other hand,

$$||T(\tau)T_h((n-1)\tau_0)u_{0h} - T_h(\tau)T_h((n-1)\tau_0)u_{0h}||_X \leq \epsilon_0/2 .$$

Finally $T_h(\tau)T_h((n-1)\tau_0)u_{0h} \in N(B_0,\epsilon_0)$, for $\tau_0 \leq \tau \leq 2\tau_0$, i.e., $T_h(t)u_{0h}$ $N(B_0,\epsilon_0)$ for $\tau_0 \leq t \leq (n+1)\tau_0$.

If the dynamical system T(t) has a local compact attractor A, the hypotheses of Lemma 2.1 can be weakened as we shall see below.

Proposition 2.2. *Assume that there exist a compact set $A \subset X$ and an open neighborhood N_1 of A such that A attracts N_1. Suppose that there are constants $h_0 > 0$, $\delta_0 > 0$, $t_0 \geq 0$ and two open neighborhoods N_2, N_3 of A, with $N_1 \subset N_2 \subset N(N_2,\delta_0) \subset N_3$, such that, for $0 < h \leq h_0$*

(i) $T(t)N_1 \subset N_2$ *for* $t \geq 0$,

(ii) $T_h(t)(N_1 \cap X_h) \subset N_2$ *for* $0 \leq t \leq t_0$,

(iii) *for any* $x_h \in N(N_2,\delta_0) \cap X_h$, *there exists* $t(x_h) > 0$ *such that* $T_h(t)x_h \in N_3$, *for* $0 \leq t \leq t(x_h)$

Also assume that $T_h(t)$ conditionally approximates $T(t)$ on N_3 uniformly on compact sets of $[t_0,+\infty)$. Then, for any $\epsilon_0 > 0$, there are $\bar{h} > 0$ and $\tau_0 > t_0$ such that, for $0 < h \leq \bar{h}$ and $t \geq \tau_0$,

$$(2.4) \qquad T_h(t)(N_1 \cap X_h) \subset N(A,\epsilon_0).$$

Proof. As $T_h(t)$ conditionally approximates $T(t)$ on N_3 uniformly on compact sets of $[t_0,+\infty)$, for any $t_1 > t_0$, there is a positive number $\bar{h}(t_1)$ so that $\eta(h,[t_0,t_1],N_3) < \delta_0/4$, for $h \leq \bar{h}(t_1)$. For any $x_h \in N_1 \cap X_h$ and any t, $t_0 \leq t \leq t_1$, we want to prove that $T_h(t)x_h \in N_3$, because this will show that $\| T(t)x_h - T_h(t)x_h \|_X \leq \eta(h,[t_0,t_1],N_3)$ for $t_0 \leq t \leq t_1$ and we may apply Lemma 2.1. Assume this is not the case. Then, by (ii) and (iii), there exists t_2, $t_0 < t_2 \leq t_1$ such that $T_h(t)x_h \in N_3$ for $0 \leq t < t_2$ and $T_h(t_2)x_h \notin N_3$. But then $T_h(t)x_h \in N(N_2,\delta_0/4)$ for $0 \leq t < t_2$ and hence $T_h(t_2)x_h \in N(N_2,\delta_0/2)$, which is a contradiction. This proves the proposition.

Remark 2.3. If A is a local, compact attractor under the semigroup T(t), then A is stable and there always exist neighborhoods N_1, N_2 satisfying (i) in Proposition 2.2.

To state the next result, we need some additional terminology. Following [Hale, LaSalle and Slemrod] (see also [Hale and Lopez]), a semigroup T(t), $t \geq 0$, on a Banach space X is said to be asympotically smooth if, for any bounded set $B \subset X$, there is a compact set $J = J(B) \subset X$ such that J attracts the set $\{x \in B: T(t)x \in B$ for $t \geq 0\}$. A special case of asymptotically smooth maps are α-contracting semigroups (see [Hale and Lopez]). In particular, T(t) is a α-contracting semigroup if $T(t) = S(t) + U(t)$ where U(t), $t \geq 0$, is completely continuous and S(t), $t \geq 0$, is a bounded linear operator for which there is a $\beta > 0$ such that $\| S(t) \|_X \leq \exp(-\beta t)$, $t \geq 0$.

The next result gives conditions for the existence of compact attractors A_h for $T_h(t)$ and the lower semicontinuity of these sets "at h = 0".

Theorem 2.4. *Assume that* T(t) *has a local, compact attractor* A *and that the hypotheses of Proposition 2.2 are satisfied. If each* $T_h(t)$ *is asymptotically smooth, then there is* $h_0 > 0$ *such that, for* $0 < h \leq h_0$, $T_h(t)$ *admits a local, compact attractor* A_h, *which attracts* $N_1 \cap X_h$. *Moreover,* $\delta_X(A_h, A) \to 0$ as $h \to 0$.

Proof. From Proposition 2.2, it follows that $T_h(t)(N_1 \cap X_h)$, $t \geq 0$, belongs to a bounded set in X_h. The results in [Hale, LaSalle and Slemrod] (see also [Hale 2]) imply the existence of a compact attractor A_h for $T_h(t)$ which attracts $N_1 \cap X_h$. Owing to Relation (2.4), we can take $A_h \subset N(A_0, \epsilon_0)$. Since ϵ_0 is arbitrary, we obtain the result.

Corollary 2.5. *Assume that* T(t) *has a local compact attractor* A *and that the conditions of Proposition 2.2 are satisfied. If each space* X_h *is finite-dimensional, the* *conclusions of Theorem 2.4 hold.*

In the general case the hypotheses of Theorem 2.4 do not enable us to give any information about the distance $\delta_X(A,A_h)$. However, if A is reduced to a point x_0, then, of course, under the hypotheses of Theorem 2.4, $\delta(A,A_h) \to 0$.

In the next result, we assume the attractors for the approximate semigroups exist and conclude that the original semigroup admits an attractor.

Proposition 2.6. *Suppose there are bounded open sets* $N_1 \subset N_2 \subset X$ *and positive constants* ϵ_0, $\bar{\epsilon}_0$, h_0, t_0, δ_0 *such that, for each* $0 < h \leq h_0$, *the semigroup* $T_h(t)$ *has a local compact attractor* $A_h \subset X$, *with* $N(A_h,\bar{\epsilon}_0) \subset N_1$ *and that*

(i) A_h *attracts* N_1 *uniformly, that is, for any* $\epsilon_1 > 0$, *there is a* $\tau_1 > 0$, *independent of* h, *such that* $T_h(t)(N_1 \cap X_h) \subset N(A_h,\epsilon_1)$ *for* $t \geq \tau_1$,

(ii) $T_h(t)(N_1 \cap X_h) \subset N_2 \cap X_h$, *for all* $t \geq 0$,

(iii) $T(t)N_1 \subset N_2$ *for* $0 \leq t \leq t_0$

(iv) $T(t)x$ *is well defined for* $x \in N(N_2,\epsilon_0)$ *for* $0 \leq t \leq \delta_0$.

Also assume that $T_h(t)$ *conditionally approximates* $T(t)$ *on* $N(N_2,\epsilon_0)$ *uniformly on compact sets of* $[t_0,+\infty)$. *Then, there exists* $\tau \geq t_0$ *such that, for* $t \geq \tau$,

(2.5) $\qquad T(t)N_1 \subset N_1$

If, in addition, $T(t)$ *is asymptotically smooth, then* $T(t)$ *has a local compact attractor* A *attracting* N_1 *and, for any* $\epsilon > 0$, *there exists* $h_1 > 0$ *such that, for* $0 < h \leq h_1$,

(2.6) $\qquad A \subset N(A_h,\epsilon)$.

Proof. Let us first show that

(2.7) $\qquad T(t)N_1 \subset N(N_2,\epsilon_0)$ for all $t \geq 0$

Owing to (iii), $T(t)N_1 \subset N(N_2,\epsilon_0)$ for $0 \leq t \leq t_0$. Suppose that the property (2.7) is not true; then there exist $x \in N_1$ and $t_2 > t_0$ such that $T(t_2)x \in \partial(\mathcal{N}(N_2,\epsilon_0))$ and $T(t)x \in \mathcal{N}(N_2,\epsilon_0)$ for $0 \leq t < t_2$ (the existence of t_2 is ensured by (iv)). Thanks to the approximation property (2.1), there exist a positive number h_2 and, for $0 < h \leq h_2$, an element $x_h \in N_1 \cap X_h$ close enough to x so that, for $0 \leq t \leq t_2$, $0 < h \leq h_2$,

$$\|T(t)x_h - T(t)x\|_X < \frac{\epsilon_0}{3} \tag{2.8}$$

Moreover, there exists $h_3 > 0$, with $h_3 \leq \inf(h_0,h_2)$ such that, for $0 < h \leq h_3$,

$$\eta(h,[t_0,t_2],\mathcal{N}(N_2,\epsilon_0)) < \frac{\epsilon_0}{3}. \tag{2.9}$$

Thus, since $T_h(t)$ conditionally approximates $T(t)$ on $\mathcal{N}(N_2,\epsilon_0)$ uniformly on $[t_0,t_2]$, (2.8), (2.9) and (ii) imply that $T(t_2)x \in Cl(\mathcal{N}(N_2,2\epsilon_0/3))$, which is a contradiction. Then (2.7) is true.

Next we show that $T(t)N_1 \subset N_1$ for $t \geq \tau$, where $\tau \geq t_0$ is a constant. Owing to the property (i), there exists $\tau \geq t_0$ such that, for $0 < h \leq h_1$, $T_h(t)(N_1 \cap X_h) \subset \mathcal{N}(A_h, \bar{\epsilon}_0/4)$ for $t \geq \tau$. *Now let* $x \in N_1$ *be given. As above, there exist a positive* number h_4, with $h_4 \leq h_0$, and, for $0 < h \leq h_4$, an element $x_h \in N_1 \cap X_h$ close to x such that, for $0 < h \leq h_4$,

$$\|T(t)x_h - T(t)x\|_X < \frac{\bar{\epsilon}_0}{4} \quad \text{for all } t, \text{ with } 0 \leq t \leq 2\tau \tag{2.10}$$

and

$$\eta(h,[t_0,2\tau], \mathcal{N}(N_2,\epsilon_0)) < \frac{\bar{\epsilon}_0}{4} \tag{2.11}$$

As (2.7) holds and $T_h(t)$ conditionally approximates $T(t)$ on $\mathcal{N}(N_2,\epsilon_0)$ uniformly on $[t_0,2\tau]$, we derive from (2.10) and (2.11) that $T(t)x \in \mathcal{N}(A_h, 3\bar{\epsilon}_0/4) \subset N_1$, for $\tau \leq t \leq 2\tau$.

An easy induction, similar to the one of the proof of lemma 2.1, shows that $T(t)x \in N_1$, for $t \geq \tau$.

If, moreover, $T(t)$ is asymptotically smooth, we conclude, by using a result of

[Cooperman] that $T(t)$ has a compact attractor $A \subset N_1$, attracting N_1. It remains to prove (2.6). Let $\epsilon > 0$ be given. By (i), there exists $\tau_1 > t_0$, independent of h, such that $T_h(t)(N_1 \cap X_h) \subset N(A_h,\epsilon/3)$ for $t \geq \tau_1$ and for $0 < h \leq h_1$. Due to the compactness of A, there exists h_δ, $0 < h_\delta \leq h_1$, such that, for $0 < h \leq h_\delta$, with each element $x \in A$ we can associate an element $P_h x$ in $N_1 \cap X_h$ such that

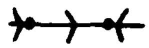
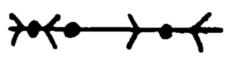
$$||T(t)x - T(t)P_h x||_X \leq \epsilon/3 \quad \text{for} \quad 0 \leq t \leq \tau_1.$$

Finally, there exists h_1, $0 < h_1 \leq h_\delta$, such that, for any x_h $\quad N_1 \cap X_h$,

$$||T(t)x_h - T_h(t)x_h||_X \leq \epsilon/3 \quad \text{for} \quad t_0 \leq t \leq \tau_1.$$

Thus, for $0 < h \leq h_1$, $T(\tau_1)x$ $\quad N(A_h,\epsilon)$, for all x in A; and from the equality $T(\tau_1)A = A$, we deduce the inclusion (2.6).

Remark 2.7. Property (2.6) means that $\delta_X(A,A_h) \to 0$ as $h \to 0$. Let us remark that, under the hypotheses of Proposition 2.6, $\delta_X(A_h, A)$ also tends to 0 as h tends to 0. Indeed, as A attracts N_1, for any $\epsilon_1 > 0$, there exists $\tau_1 > t_0$ such that $T(t)A_h \subset N(A,\epsilon_1/2)$ for $t \geq t_1$. On the other hand, there exists $\bar{h} > 0$ such that, for $0 < h \leq \bar{h}$, $\eta(h,[t_0,\tau_1],N(N_2,\epsilon_0)) \leq \epsilon_1/2$. Thus, $T_h(t_1)A_h \subset N(A,\epsilon_1)$ and, since $T_h(t_1)A_h = A_h$, $A_h \subset N(A,\epsilon_1)$, for $0 < h \leq \bar{h}$.

Remark 2.8. The assumption (i) in Proposition 2.6 that A_h attracts N_1 uniformly is a very strong condition. However, one would expect numerical procedures to have such a property. The detailed structure of the flow on the attractor A_h could vary considerably with h. This depends on the flow defined by T(t). Consider, for example a scalar equation $\dot{u} = f(u)$, $u \in R$, where the flow is given by .
If one approximates this flow numerically, two situations could arise. One could obtain either the approximate flow  or . The global attrractor in one case is a point and in the other is a line segment. The

global attractor for the original problem is a line segment. For one of the approximation schemes, the attactors A_h approach a point as $h \to 0$ which is a local attractor for $T(t)$ and, for the other, A_h approaches a line segment which is the global attractor for $T(t)$. If the flow on the attractor for $T(t)$ is less sensitive to small perturbations, this situation will not arise.

Let us now turn to the question of how close A_h is to A with the measure of closeness given by $\delta_X(A_h,A)$. We give some results in this direction for some particular cases.

Proposition 2.9. *Suppose the hypotheses of Theorem 2.4 are satisfied with the associated function* $\eta(h,I,N_3) = ch^{\gamma_0}$ *for some positive constants* c, γ_0, *independent of* h *and* $I \subset [t_0,\infty)$. *Then there is a constant* $c_1 > 0$ *such that* $\delta_X(A_h,A) \leq c_1 h^{\gamma_0}$ *for* $0 < h \leq h_0$.

Proof. The proof follows from the proof of Proposition 2.2 and Theorem 2.4 using the special function $\eta(h,I,N_3) = ch^{\gamma_0}$.

The hypothesis on $\eta(h,I,N_3)$ in Proposition 2.9 is not usually satisfied. A more reasonable condition on $\eta(h,I,N_3)$ is given in the next result, but then we must impose stronger attractivity properties of A.

Proposition 2.10. *Assume the hypotheses of Theorem 2.4 are satisfied with the associated function* $\eta(t,[t_0,t_1],N_3) = c_0 h^{\gamma_0} e^{\alpha_0 t_1}$ *for some positive constants* c_0, γ_0, α_0, *independent of* h *and* t_1. *If there are an open neighborhood* U *of* A *and positive constants* c_1, β_0 *such that*

$$\delta_X(T(t)U,A) \leq c_1 e^{-\beta_0 t}, \ t \geq 0,$$

then, for $h \leq h_0$, *we have*

$$\delta_X(A_h, A) \leq ch^{\gamma_0\beta_0/(\alpha_0+\beta_0)}$$

for some positive constant c.

Proof. If

$$t_1 = -\frac{1}{\beta_0} \log \frac{c_0}{c_1} h^{\gamma_0\beta_0/(\alpha_0+\beta_0)},$$

then $\delta_X(T(t)U, A) \leq c_0 h^{\gamma_0\beta_0/(\alpha_0+\beta_0)}$ for $t \geq t_1$. Since A_h is invariant, for any $x_h \in A_h$, there is a $y_h \in A_h$ such that $x_h = T_h(t_1)y_h$. If $x = T(t_1)y_h$, then

$$||x_h - x||_X = ||T_h(t_1)y_h - T(t_1)y_h||_X \leq c_0^{1-\alpha_0/\beta_0} c_1 h^{\gamma_0\beta_0/(\alpha_0+\beta_0)}$$

This completes the proof.

Remark 2.11. If T(t) is a gradient system (for the definition, see [Hale, 3) for

which there is a $t_1 > 0$ such that T(t) is either compact for $t > t_1$ or an α-contraction, and if the set of equilibrium points E (i.e. the points x such that $T(t)x = x$, $t \geq 0$) is bounded, then we know that T(t) has a compact attractor A. If, in addition, each element of E is hyperbolic, then E is a finite set, $\dim W^u(\Phi) < +\infty$ and $A = \bigcup_{\phi\in E} W^u(\phi)$ where $W^u(\phi)$ is the unstable set of ϕ. Furthermore, there is an open neighborhood U of A such that $\delta_X(T(t)U, A) \to 0$ exponentially as $t \to +\infty$.

Thus, if the approximate semigroups $T_h(t)$ satisfy the hypotheses of Theorem 2.4 with $\eta(h,[t_0,t_1),N_3) = c_0 h^{\gamma_0} e^{\alpha_0 t}$, $T_h(t)$ admits a local compact attractor A_h for h small enough and, by Proposition 2.10, we obtain a good estimate of $\delta(A_h, A)$.

Now assume that, for $h > 0$, $T_h(t)$ is a gradient system. Then, one can prove that, for h small enough, the set of equilibrium points E_h of T_h is finite and has the same cardinality as E and one can give an estimate of $\delta_X(E, E_h)$ and $\delta_X(E_h, E)$. Moreover $A_h = \bigcup_{\phi_h\in E_h} W^u(\phi_h)$ where $W^u(\phi_h)$ is the unstable set of ϕ_h. (For more details, see [Lin and Raugel]).

In Remark 2.11, we have encountered a situation where the conditions of Proposition 2.10 are satisfied. One would expect that the hypothesis in Proposition 2.10 that $T(t)U \to \mathcal{A}$ exponentially as $t \to +\infty$ will be satisfied in specific evolutionary problems at least generically with respect to the vector fields. A more precise statement is needed and certainly is nontrivial.

Let us end this section by pointing out that in some cases the semigroups $T_h(t)$ do not conditionally approximate $T(t)$ on any open set $V \subset X$. In this case, one has to use other ways to prove that $T_h(t)$ admits a local compact attractor $\mathcal{A}_h$ for h small enough. In section 7.2 we shall encounter a typical example of this case.

3. Approximation of Sectorial Evolutionary Equations with Special Projection Methods.

Let A be a sectorial (linear) operator on a Banach space X. We recall that A is sectorial if and only if the semigroup e^{-At} generated by A is an analytic semigroup; and if A is a sectorial operator on X with $Re\sigma(A)$, where $\sigma(A)$ denotes the spectrum of A, then, for any $\alpha > 0$, one can define the operators $A^{-\alpha}$ and A^{α}. Moreover, if $Re\sigma(A) > \lambda > 0$, for any $\alpha \geq 0$, there exists a constant $c_2 < +\infty$ such that

$$\|A^{\alpha}e^{-At}\|_{L(X,X)} \leq C_{\alpha}t^{-\alpha}e^{-\lambda t}, \; t > 0.$$

If A is a sectorial ooperator on X, then there is a real number $a > 0$ such that $A_1 = A + aI$ satisfies $Re\sigma(A_1) > 0$. If we define $X^{\alpha} = D(A_1^{\alpha})$, $\alpha \geq 0$, with the graph norm $\|x\|_{X^{\alpha}} = \|A_1^{\alpha}x\|_X$, $x \in X^{\alpha}$, then X^{α} is a Banach space normed by $\|\cdot\|_{X^{\alpha}}$ (for more details, see [Henry, p. 26-29]).

Now we consider the nonlinear equation

$$(3.1) \qquad \begin{cases} \dfrac{du}{dt} + Au = f(u), \\ \\ u(0) = u_0, \end{cases}$$

where there exists a real number $\alpha \in [0,1]$ such that $f: X^{\alpha} \to X$ is locally Lipschitz continuous (i.e. f is continuous and, for any bounded set U in X^{α}, there is a constant k_u such that, $\|f(u) - f(v)\|_X \leq k_u\|u - v\|_{X^{\alpha}}$, for u,v in V).

A solution of (3.1) on $[0,\tau)$ is a continuous function $u : [0,\tau) \to X^{\alpha}$, $u(0) = u_0$, which satisfies the relation

$$(3.2) \qquad u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}f(u(s))ds, \;\; 0 \leq t < \tau.$$

One can prove (see [Henry, p. 54-57, 62-65]) that, under the above hypotheses on A, f, there is a unique solution of (3.1) on a maximal interval of existence $[0,\tau_{u_0})$. If, in addition, f is a C^r-function in u, the solution $u(t,u_0)$ is a C^r-function in (t,u_0) on $[0,\tau_{u_0})$.

Here we assume that all solutions are defined for $t \geq 0$ so that we can introduce the map $T(t) ; X^\alpha \to X^\alpha$, $t \geq 0$, defined by $T(t)u_0 = u(t,u_0)$ and obtain a C^r-seimgroup on X with $r \geq 0$ (we also suppose that T(t) has a local compact attractor $\mathcal{A}$ which attracts an open set $\mathcal{O} \supset \mathcal{A}$ (see [Hale 3] for the existence of $\mathcal{A}$).

Remark 3.1. We may always assume that $\mathrm{Re}\sigma(A) > \lambda > 0$. Indeed, as A is a sectorial operator, there exists a positive number a such that, if $A_1 = A + aI$, $\mathrm{Re}\sigma(A_1) > \lambda > 0$. Then we replace equation (3.1) by

$$
(3.1) \qquad \begin{cases} \dfrac{du}{dt} + A_1 u = f(u) + aI \\ \\ u(0) = u_0 \end{cases}
$$

Therefore, we suppose in the sequel that $\mathrm{Re}\sigma(A) > \lambda > 0$. We assume also that $\sigma(A)$ consists of isolated points λ_n only with no accumulation in the finite part of $\mathbf{C}$ (i.e. ∞ is the only possible accumulation point) and that each λ_n is of finite order. We order the points λ_n in such a way that

$$
\lambda < \mathrm{Re}\lambda_1 \leq \mathrm{Re}\lambda_2 \leq \ldots \leq \mathrm{Re}\lambda_n \leq \mathrm{Re}\lambda_{n+1} \leq \ldots,
$$

where $\mathrm{Re}\lambda_n \to +\infty$ as $n \to +\infty$.

We denote by Φ_n the generalized eigenspace corresponding to λ_n, by P_N the projection from X onto the space $[\Phi_1,\Phi_2,\ldots,\Phi_N]$ and by Q_N the projection $I - P_N$. We assume that, for $0 \leq \beta < 1$, $\|P_N\|_{\mathcal{L}(X^\beta;X^\beta)}$ is bounded by a constant $K_\beta > 0$, uniformly with respect to N. By [Henry, p.21], for any $\epsilon > 0$, for any integer N, there exists a constant $K_{\epsilon,N}$ such that

$$(3.3) \qquad \|A^j e^{-At}Q_N\|_{\mathcal{L}(X,X)} \leq K_{\epsilon,N} \frac{e^{-(Re\lambda_{N+1}-\epsilon)t}}{t^j}, \quad \text{for } j = 0,1;$$

Below, we assume that, for $0 \leq \beta < 1$,

$$(3.4) \qquad \lim_{N\to+\infty} \frac{K_{\epsilon,N}}{(Re\lambda_{N+1}-\epsilon)^{\beta}} = 0 ,$$

this condition being usually satisfied.

Now let us consider the following equation on $X_N = P_N X$:

$$(3.5)_N \qquad \begin{cases} \dfrac{du_N}{dt} + Au_N = P_N f(u_N), \\ \\ u_N(0) = u_{0N} , \end{cases}$$

where $u_{0N} \in X_N$. Equation $(3.5)_N$ is an ordinary differential equation. Let us introduce the map $T_N(t) : X_N \to X_N$, defined by $T_N(t)u_{0N} = u_N(t,u_{0N})$, as long as $u_N(t,u_{0N})$ exists. $T_N(t)u_{0N}$ is continuous in t and u_{0N}, when it is well defined and, if $T_N(t)u_{0N}$ exists on $[t_0,t_1)$, it is left-continuous at t_1.

Theorem 3.1. *Under the above hypotheses, there exists a number* $N_0 > 0$ *such that, for* $N \geq N_0$, $T_N(t)$ *admits a local compcat attractor* $\mathcal{A}_N$ *which attracts an open set* $\mathcal{N}_1 \cap X_N$, *where* $\mathcal{N}_1$ *is independent of* N. *Moreover,* $\delta_X(\mathcal{A}_N,\mathcal{A}) \to 0$ as $N \to +\infty$.

Proof. Clearly, Theorem 3.1 is proven, thanks to Corollary 2.5 and Proposition2.2, if we show that there are constants $\delta_0 > 0$, $N_0 > 0$, and three open neighborhoods $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ of $\mathcal{A}$ such that $\mathcal{N}_1 \subset \mathcal{N}_2 \subset \mathcal{N}(\mathcal{N}_2,\delta_0) \subset \mathcal{N}_3$, $\mathcal{N}_1 \subset \mathcal{O}$, and

(i) $T(t)\mathcal{N}_1 \subset \mathcal{N}_2$ for $t \geq 0$,

(ii) for $N \geq N_0$, for any $t_1 > 0$, if $T(t)u_{0N}$ and $T_N(t)u_{0N}$ belong to $\mathcal{N}_3$ for $0 \leq t \leq \tau$, with $\tau \leq t_1$, then, for $0 \leq t \leq \tau$,

$$(3.6)\qquad \|T_N(t)u_{0N} - T(t)u_{0N}\|_{X^\alpha} \leq \eta(N,[0,t_1],\mathcal{N}_3)$$

with

$$(3.7)\qquad \lim_{N\to+\infty} \eta(N,[0,t_1],\mathcal{N}_3) = 0.$$

As $\mathcal{A}$ is a compact attractor, there are two open neighborhoods $\mathcal{N}_1,\mathcal{N}_2$ of $\mathcal{A}$ such that (i) holds and $\mathcal{N}_1 \subset \mathcal{O}$. Let δ_0 be a positive real number and set $\mathcal{N}_3 = \mathcal{N}(\mathcal{N}_2,\delta_0)$. As f is locally Lipschitz continuous, there exist two constants $M_1 > 0$ and $L > 0$ such that

$$(3.8)\qquad \forall\ u,\ v \in \mathcal{N}(\mathcal{N}_3,\delta_0),\ \|f(u) - f(v)\|_X \leq L\|u-v\|_{X^\alpha},$$

and

$$(3.9)\qquad \forall\ v \in \mathcal{N}(\mathcal{N}_3,\delta_0),\ \|f(v)\|_X \leq M_1.$$

Now it remains to prove the property (ii); to this end, we assume that $u(t) = T(t)u_{0N}$ and $u_N(t) = T_N(t)u_{0N}$ belong to $\mathcal{N}_3$. At first, we compare u(t) with its projection $\tilde{u}_N(t) = P_N u(t)$. As P_N and A commute, we have:

$$\tilde{u}_N(t) = e^{-At}\ u_{0N} + \int_0^t e^{-A(t-s)}P_N f(u(s))ds,$$

and therefore, by (3.2),

$$u(t) - \tilde{u}_N(t) = \int_0^t e^{-A(t-s)}Q_N f(u(s))\,ds.$$

Using (3.3) and [Henry, Theorem 1.4.3, page 26], we get

$$\|e^{-At}\,Q_N\|_{\mathcal{L}(X;X^\alpha)} \leq K_{\epsilon,N}\, e^{-(\mathrm{Re}\lambda_{N+1}\,-\,\epsilon)t}\ \Gamma(\alpha)t^{-\alpha},$$

and therefore, thanks to (3.9), we have

$$||u(t) - \tilde{u}_N(t)||_{X^\alpha} \leq M_1 K_{\epsilon,N} \Gamma(\alpha) \int_0^\infty \sigma^{-\alpha} e^{-(\mathrm{Re}\lambda_{N+1} - \epsilon)\sigma} \, d\sigma$$

or

$$(3.10) \qquad ||u(t) - \tilde{u}_N(t)||_{X^\alpha} \leq \frac{M_1 K_{\epsilon,N} \Gamma(\alpha)\Gamma(1-\alpha)}{(\mathrm{Re}\lambda_{N+1} - \epsilon)^{1-\alpha}}$$

Therefore

$$(3.11) \qquad ||u(t) - \tilde{u}_N(t)||_{X^\alpha} \leq \epsilon_N ,$$

where ϵ_N does not depend on t and $\lim_{N \to +\infty} \epsilon_N = 0$. Hence, for $N \geq N_0$, $\tilde{u}_N(t) \in \mathcal{N}(\mathcal{N}_3, \delta_0)$ as soon as $u(t) \in \mathcal{N}_3$.

Now we compare $u_N(t)$ with $\tilde{u}_N(t)$. We have:

$$u_N(t) - \tilde{u}_N(t) = \int_0^t e^{-A(t-s)}(P_N f(u_N(s)) - P_N f(u(s)))ds$$

and hence

$$||u_N(t) - \tilde{u}_N(t)||_{X^\alpha} \leq K_0 C_\alpha \int_0^t e^{-\lambda(t-s)}(t-s)^{-\alpha}(||f(u(s)) - f(\tilde{u}_N(x))||_X$$

$$+ ||f(u_N(s)) - f(\tilde{u}_N(s))||_X)$$

or also, by (3.8),

$$||u_N(t) - \tilde{u}_N(t)||_{X^\alpha} \leq K_0 L C_\alpha \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)}(||u_N(s) - \tilde{u}_N(s)||_{X^0} + \epsilon_N)ds.$$

Let us set: $w(t) = (\tilde{u}_N(t) - u_N(t))e^{\lambda t}$. Then we get

$$(3.12) \qquad ||w(t)||_{X^\alpha} \leq K_0 L\, C_\alpha \int_0^t (t-s)^{-\alpha} e^{\lambda s} \epsilon_N ds$$

$$+ K_0 L C_\alpha \int_0^t (t-s)^{-\alpha} ||w(s)||_{X^\alpha} ds$$

Using a more general form of Gronwall's lemma (see [Henry, page 6]), we deduce from (3.12) that

$$(3.13) \qquad \|w(t)\|_{X^\alpha} \leq \epsilon_N K_0 L C_\alpha e^{\lambda t} \frac{t^{1-\alpha}}{1-\alpha} M(t_1) ,$$

where $M(t_1)$ is independent of N and is an increasing function of t_1. From (3.13), we derive:

$$(3.14) \qquad \|u_N(t) - \tilde{u}_N(t)\|_{X^\alpha} \leq \epsilon_N \tilde{M}(t_1) ,$$

where $\tilde{M}(t_1) > 0$ is independent on N and is an increasing function of t_1. The estimates (3.11) and (3.14) give us the conditions (3.6) and (3.7).

4. Galerkin Approximation of Some Parabolic Equations

4.1. A general result

Let V and H be two (real) Hilbert spaces such that V is included in H with a continuous and dense imbedding; the space H is identified with its dual space, and the inner product of H, as well as the duality pairing between V and its dual space V' is denoted by $(\cdot,\cdot)$ (so we have the inclusions $V \subset H \subset V'$ where the imbeddings are continuous and dense). We introduce a continuous, bilinear form on $V \times V$: $(u,v) \in V \times V \to a(u,v)$ and the corresponding operator $A \in \mathcal{L}(V;V')$ defined by

$$\forall\, u,\ v \in V, \quad a(u,v) = (Au,v).$$

We denote by C_0 the constant of continuity of the bilinear form $a(\cdot,\cdot)$. We also suppose that there are two constants $\gamma > 0$ and $\gamma_0 > 0$ such that

$$\forall\, v \in V, \quad a(v,v) + \gamma_0 \|v\|_H^2 \geq \gamma \|v\|_V^2 . \tag{4.1}$$

Moreover, if

$$b(u,v) = a(u,v) - a(v,u),$$

we assume that there exists a constant $C_1 > 0$ such that

$$|b(u,v)| \leq C_1 \|u\|_V \|v\|_H . \tag{4.2}$$

Now we consider the nonlinear equation

$$\begin{cases} \dfrac{du}{dt} + Au = f(u), \\ \\ u(0) = u_0, \end{cases} \tag{4.3}$$

where $f : V \to H$ is locally Lipschitz continuous and $u_0 \in V$.

If $D(A) = \{v \in V; Av \in H\}$, then $D(A)$ is dense in V and in H, and A is a sectorial operator on H so that we can define the operators A^α, $0 \leq \alpha \leq 1$. If

$$(4.4) \qquad D(A^{1/2}) = D(A^{*1/2}) = V,$$

where A^* is the adjoint operator of A, defined by

$$\forall u, v \in V, \quad (A^*u,v) = a(v,u),$$

we are in the frame given in Section 3; therefore, if we assume that all solutions are defined for $t \geq 0$, we can introduce the map $T(t) : V \to V$, $t \geq 0$, defined by $T(t)u_0 = u(t,u_0)$ and obtain a C^0-semigroup on V. [Here we also assume that T(t) has a local compact attractor $\mathcal{A}$ which attreacts a bounded open set $\mathcal{O}$, $\mathcal{O} \supset \mathcal{A}$.

Remark 4.1: We may always assume that $\gamma_0 = 0$. If $\gamma_0 > 0$, we can set $A_1 = A + \gamma_0 I$ and replace equation (4.4) by

$$(4.3)' \qquad \begin{cases} \dfrac{du}{dt} + A_1 u = f(u) + \gamma_0 u \\ \\ u(0) = u_0. \end{cases}$$

Therefore we assume in the sequel that $\gamma_0 = 0$.

Remark 4.2: Condition (4.2) is satisfied if, for instance $D(A) = D(A^*)$, which is true, in particular, if A is an elliptic differential operator, with Dirichlet boundary conditions, the data being sufficiently regular (see [Lions] or [Kato]).

Now let us turn turn to a finite-dimensional approximation of equation (4.3). Let $h > 0$ be a real parameter which will tend to 0 and $(V_h)_h$ a family of finite-dimensional subspaces of V. We introduce the operator $A_h \in \mathcal{L}(V_h;V_h)$ defined by

(4.5) $\quad \forall\, v_h \in V_h,\ (A_h w_h, v_h) = a(w_h, v_h)$ for w_h in V_h.

Let $Q_h \in \mathcal{L}(H;V_h)$ be the projector on V_h in the space H, i.e.

$$\forall v \in H,\ \forall v_h \in V_h,\ (v - Q_h v, v_h) = 0,$$

and let $P_h \in \mathcal{L}(V;V_h)$ be the projector on V_h in the space V, i.e.

$$\forall v \in V,\ \forall v_h \in V_h,\ a(v - P_h v, v_h) = 0.$$

Now let us consider the following equation in V_h:

$$(4.3)_h \qquad \begin{cases} \dfrac{du_h}{dt} + A_h u_h = Q_h f(u_h), \\ \\ u_h(0) = u_{oh} \end{cases}$$

where $u_{oh} \in X_h$. Equation $(4.3)_h$ is an ordinary differential equation. Let us introduce the map $T_h(t) : V_h \to V_h$, defined by $T_h(t)u_{oh} = u_h(t, u_{oh})$ as long as $u_h(t,u_{oh})$ exists. $T_h(t)u_{oh}$ is continuous in t and u_{oh} when it is well defined and, if $T_h(t)u_{oh}$ exists on $[t_0,t_1)$, it is left-continuous at t_1.

In order to prove that $T_h(t)$ also admits a compact attractor $\mathcal{A}_h$, for h small enough, we need the following additional hypotheses on the spaces $(V_h)_h$:

- there exists a constant $m > 0$ and, for any β, $1/2 \leq \beta \leq 1$, a constant $C(\beta) > 0$ such that, for all w in $D(A^\beta)$,

(4.6)(i) $\quad ||w - P_h w||_V + ||w - Q_h w||_V \leq C(\beta) h^{2m(\beta - 1/2)} ||w||_{X^\beta}$

and

(4.6)(ii) $\quad ||w - P_h w||_H + ||w - Q_h w||_H \leq C(\beta) h^{2m\beta} ||w||_{X^\beta}$,

where $X^\beta = D(A^\beta)$ and $D(A) = \{v \in V : Av \in H\}$

The hypotheses (4.6)(i) and (4.6)(ii) are realistic and are satisfied in many cases, when A is an elliptic differential operator [Ciarlet] and also the example 4.1 below.

Example 4.1. Let Ω be a regular bounded domain on a convex bounded set in $\mathbf{R}^2$. In Ω we are given an elliptic operator of the following form:

$$(4.7) \qquad Lv = \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{j=1}^{2} b_j(x) \frac{\partial v}{\partial x_j} + c(x)v \;,$$

where the coefficients a_{ij}, b_j, c are smooth enough and where L is assumed to be uniformly and strongly elliptic. If A denotes the operator $-L$, with homogeneous Dirichlet boundary conditions, then the hypotheses (4.1),(4.2) and (4.4) are satisfied with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $D(A^{1/2}) = V = H_0^1(\Omega)$, $H = L^2(\Omega)$. And one can find finite dimensional subspaces V_h of $H_0^1(\Omega)$ such that the conditions (4.5)(i), (4.6)(ii) are satisfied with $m = 1$. For instance, if Ω is a convex polygonal domain, we introduce a uniformly regular family $(\mathcal{T}_h)_h$ of triangulations in the sense of [Ciarlet] where $\mathcal{T}_h$ is made of traingles with diameters bounded by h. And we set:

$$(4.8) \qquad V_h = \{v_h \in C^0(\bar{\Omega}) \cap H_0^1(\Omega) : \forall\, K \in \mathcal{T}_h,\ v_{h/k} \in P_1(k)\}$$

where $P_1(k)$ is the space of all polynomials of degree ≤ 1 on K. In this case, the hypotheses (4.6)(i) and (4.6(ii) are satisfeid with $m = 1$. Moreover, even if the family $(\mathcal{T}_h)_h$ is only regular, the hypothesis (4.6)(ii) is satisfied and the condition (4.6)(i) usually holds (see [Crouzeix-Thomee]).

Theorem 4.1. *Under the above hypotheses, there exists* $h_0 > 0$ *such that, for* $h \leq h_0$, $T_h(t)$ *admits a local compact attractor* $\mathcal{A}_h$, *which attracts an open set* $N_1 \cap V_h$ *where* N_1 *is independent of* h. *Moreover,* $\delta_V(\mathcal{A}_h, \mathcal{A}) \to 0$ *as* $h \to 0$.

Proof. We shall prove that the hypotheses of Proposition 2.2 are satisfied by $T(t)$ and $T_h(t)$ for h small enough. Clearly, it is sufficient to show that there are constants $h_0 > 0$, $\delta_0 > 0$ and $t_0 > 0$ and three open neighborhoods N_1, N_2, N_3, of $\mathcal{A}$ with $N_1 \subset \mathcal{O}$, $N_1 \subset N_2 \subset \mathcal{N}(N_2,\delta_0) \subset N_3$, such that the conditions (i) and (ii) of Proposition 2.2 are satisfied and that $T_h(t)$ conditionally approximates $T(t)$ on N_3 uniformly on compact sets of $[t_0,+\infty)$. Let us prove it in three steps.

First step. As $\mathcal{A}$ is a compact attractor, there is a bounded open neighborhood N_1 of $\mathcal{A}$ such that $N_1 \subset \mathcal{O}$ and $T(t)N_1 \subset N_1$, for $t \geq 0$. We choose a real number $\epsilon_0 \geq \frac{8B_0C_0}{\gamma}$ where $B = \max\limits_{v\ N_1}||v||_V$ and we set: $N_2 = \mathcal{N}(N_1,\epsilon_0)$. Finally, let δ_0 be a positive real number and define $N_3 = \mathcal{N}(N_2,\delta_0)$. Now we want to prove that there exists a constant $t_0 > 0$ such that $T_h(t)(N_1 \cap X_h) \subset N_2$ for $0 \leq t \leq t_0$. Using classical arguments of the theory of differential equations, we easily see that it is sufficient to prove the following property:

$$
\text{(A)} \quad \begin{cases} \text{there exists a constant } t_0 > 0 \text{ independent of } h \text{ such that, for any} \\ u_{0h} \in N_1 \cap V_h, \text{ if } T_h(t)u_{oh} \text{ belongs to } \mathcal{N}(u_{oh},\epsilon_0+\delta_0) \text{ for } 0 \leq t \leq t(u_{oh}), \\ \text{where } t(u_{oh}) \leq t_0, \text{ then } T_h(t)u_{oh} \in \mathcal{N}(u_{oh},\epsilon_0) \text{ for } 0 \leq t \leq t(u_{oh}). \end{cases}
$$

As f is globally Lipschitz continuous on $\mathcal{N}(N_3,\delta_0)$, there exist constants $M_1 > 0$ and $L > 0$ such that

$$\text{(4.9)(i)} \quad \forall v \in \mathcal{N}(N_3,\delta_0),\ ||f(v)||_H \leq M_1,$$

and

$$\text{(4.9)(ii)} \quad \forall v,w \in \mathcal{N}(N_3,\delta_0),\ ||f(v) - f(w)||_H \leq L||v-w||_V.$$

If u_h is the solution of Equation $(4.3)_h$, $u_h - u_{oh}$ satisfies the equation

$$\text{(4.10)} \quad \frac{d}{dt}(u_h - u_{oh}) + A_h(u_h - u_{oh}) = Q_h f(u_h) + A_h u_{oh}.$$

Taking the inner product in H of the equation (4.10) by $\frac{d}{dt}(u_h - u_{oh})$, we obtain:

$$(4.11)\qquad ||\frac{d}{dt}(u_h - u_{oh})||_H^2 + a(u_h - u_{oh}, \frac{d}{dt}(u_h - u_{oh}))$$

$$= (f(u_h), \frac{d}{dt}(u_h - u_{oh})) + \frac{d}{dt}\, a(u_{oh}, u_h - u_{oh})$$

But

$$(4.12)\qquad a(u_h - u_{oh}, \frac{d}{dt}(u_h - u_{oh})) = \frac{1}{2}\frac{d}{dt} a(u_h - u_{oh}, u_h - u_{oh})$$

$$+ \frac{1}{2}\, b(u_h - u_{oh}, \frac{d}{dt}(u_h - u_{oh})),$$

so that we deduce from (4.11) and (4.12), by using the inequality (4.2), that

$$||\frac{d}{dt}(u_h - u_{oh})||_H^2 + \frac{1}{2}\frac{d}{dt} a(u_h - u_{oh}, u_h - u_{oh}) \leq M_1 ||\frac{d}{dt}(u_h - u_{oh})||_H$$

$$+ C_1 ||u_h - u_{oh}||_V ||\frac{d}{dt}(u_h - u_{oh})||_H$$

$$+ \frac{d}{dt} a(u_{oh}, u_h - u_{oh}),$$

which implies that

$$(4.13)\qquad \frac{d}{dt} a(u_h - u_{oh}, u_h - u_{oh}) \leq M_1^2 + C_1^2 ||u_h - u_{oh}||_V^2$$

$$+ 2\frac{d}{dt}\, a(u_{oh}, u_h - u_{oh}).$$

Finally, integrating (4.13) from 0 to t_h and using (4.1) (with $\gamma_0 = 0$) and the inequality $ab \leq \frac{1}{2\epsilon} a^2 + \frac{\epsilon}{2} b^2$, we obtain:

$$||u_h(t_h) - u_{oh}||_V^2 \leq \frac{2t_h M_1^2}{\gamma} + \frac{2C_1^2}{\gamma}\int_0^{t_h} ||u_h(s) - u_{oh}||_V^2\, ds + \frac{4C_0^2}{\gamma^2}||u_{oh}||_V^2.$$

Thanks to Gronwall's inequality, we derive from the above estimate that

$$(4.14)\qquad ||u_h(t_h) - u_{oh}||_V^2 \leq \left[\frac{2t_h M_1^2}{\gamma} + \frac{4C_0^2}{\gamma^2}||u_{oh}||_V^2\right] e^{\frac{2c_1^2}{\gamma_0} t_h}$$

If $u_{oh} \in N_1 \cap V_h$, (4.14) becomes

$$\|u_h(t_h) - u_{oh}\|_V^2 \leq \left[\frac{2t_h M_1^2}{\gamma} + \frac{4C_0^2}{\gamma^2} B_0^2\right] e^{\frac{2c_1^2}{\gamma_0} t_h} \tag{4.15}$$

From (4.15), it is clear that there exists a constant $t_0 > 0$ independent of h such that Property (A) holds.

It remains to prove that $T_h(t)$ conditionally approximates $T(t)$ on N_3 uniformly on compact sets of $[t_0,+\infty)$. To this end, we begin by an estimate of $\|T_h(t)u_{oh} - T(t)u_{oh}\|_H$.

Step 2. Estimate of $\|T_h(t)u_{oh} - T(t)u_{oh}\|_H$ for $0 \leq t \leq t_1$, when $T_h(\tau)u_{oh}$ and $T(\tau)u_{oh}$ belong to N_3 for $0 \leq \tau \leq t$. We set $u(t) = T(t)u_{oh}$ and $u_h(t) = T_h(t)u_{oh}$. Let us remark that

$$\|T_h(t)u_{oh} - T(t)u_{oh}\|_H \leq \|u(t) - Q_h u(t)\|_H + \|Q_h u(t) - u_h(t)\|_H \tag{4.16}$$

Thanks to the hypothesis (4.6)(ii), we have

$$\|u(t) - Q_h u(t)\|_H \leq C(\tfrac{1}{2})h^m\|u(t)\|_V , \tag{4.17}$$

and it remains to estimate $\|Q_h u(t) - u_h(t)\|_H$. The function $Q_h u - u_h$ satisfies the equation

$$\begin{cases} \dfrac{d}{dt}(Q_h u - u_h) + A_h(Q_h u - u_h) = Q_h f(u) - Q_h f(u_h) + (A_h Q_h - Q_h A)u , \\ (Q_h u - u_h)(0) = 0 . \end{cases} \tag{4.18}$$

Taking the inner product in H of (4.18) by $Q_h u - u_h$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_h - Q_h u\|_H^2 &+ a(u_h - Q_h u, u_h - Q_h u) \\ &= (f(u) - f(u_h), u_h - Q_h u) + a(u - Q_h u, u_h - Q_h u) \end{aligned} \tag{4.19}$$

which implies the inequality

$$\frac{1}{2}\frac{d}{dt}||u_h - Q_h u||_H^2 + \gamma||u_h - Q_h u||_V^2 \leq L||u - Q_h u||_V||u - Q_h u||_H$$

$$+ L||u_h - Q_h u||_V||u_h - Q_h u||_H$$

$$+ C_0||u - Q_h u||_V||u_h - Q_h u||_V$$

From the above estimate, we infer (after an integration from 0 to t):

$$||u_h(t) - Q_h u(t)||_H^2 \leq \left[L^2 + \frac{4L^2}{\gamma}\right]\int_0^t ||u_h(s) - Q_h u(s)||_H^2\, ds$$

$$+ \left[L^2 + \frac{4C_0^2}{\gamma}\right]\int_0^t ||u(s) - Q_h u(s)||_V^2 ds.$$

Using Gronwall's lemma, we finally obtain

$$(4.20) \qquad ||u_h(t) - Q_h u(t)||_H \leq c^* e^{ct}\left[\int_0^t ||u(s) - Q_h u(s)||_V^2\, ds\right]^{1/2}$$

where c and c^* are two positive constants independent of h. Due to the hypothesis (4.6)(i), we have:

$$(4.21) \qquad \left[\int_0^t ||u(s) - Q_h u(s)||_V^2 ds\right]^{1/2} \leq C(\tfrac{1}{2})h^m\left[\int_0^t ||Au(s)||_H^2\, ds\right]^{1/2}.$$

Since $\frac{du}{ds}(s)$ belongs to H for s > 0, we may consider the inner product in H of Equation (4.3) by $\frac{du}{ds}$; then we get, by using a relation similar to (4.12):

$$||\frac{du}{dt}||_H^2 + \frac{1}{2}\frac{d}{dt}a(u,u) \leq ||f(u)||_H||\frac{du}{dt}||_H + \frac{c_1}{2}||u||_V||\frac{du}{dt}||_H\,,$$

and also

$$(4.22) \qquad \int_0^t ||\frac{du}{dt}||_H^2\, ds \leq 2\int_0^t ||f(u)||_H^2\, ds + 2C_1^2\int_0^t ||u||_V^2 ds + c_0||u(t)||_V^2$$

Since $Au = f(u) - \frac{du}{dt}$, we deduce from (4.22) that there exists a constant $c_0(N_2,\delta_0) > 0$, depending on N_2 and δ_0 only such that

$$(4.23) \qquad \int_0^t ||Au||_H^2 \, ds \leq c_0(N_2,\delta_0)(1 + t_1).$$

Finally, we derive from (4.16), (4.17), (4.20) amd (4.23) that, for $0 \leq t \leq t_1$,

$$(4.24) \qquad ||u(t) - u_h(t)||_H \leq C_1(N_2,\delta_0)(1+t_1)e^{ct_1}h^m,$$

where $C_1(N_2,\delta_0)$ is a positive constant depending on N_2 and δ_0 only.

Step 3. <u>Estimate of</u> $||T_h(t)u_{oh} - T(t)u_{oh}||_V$ <u>for</u> $t_0 \leq t \leq t_1$, <u>when</u> $T_h(\tau)u_{oh}$ <u>and</u> $T(\tau)u_{oh}$ <u>belong to</u> N_3 <u>for</u> $0 \leq \tau \leq t$.

To this end we at first estimate the term $||\tau T_h(\tau)u_{oh} - \tau T(\tau)u_{oh}||_V$ for $0 \leq \tau \leq t$. Let us set $Z(\tau) = \tau u(\tau)$. As $Q_hA = A_hP_h$, the function $P_hZ(\tau) = \tau P_hu(\tau)$ satisfies the equation

$$\frac{d}{dt} P_hZ + A_hP_hZ = \tau Q_hf(u) + \tau\left(\frac{d}{dt}(P_hu - Q_hu)\right) + P_hu .$$

Hence, $Z_h(\tau) - P_hZ(\tau)$ satisfies the equation

$$(4.25) \qquad \begin{aligned} \frac{d}{dt}(Z_h - P_hZ) + A_h(Z_h - P_hZ) = {} & \tau Q_h(f(u_h) - f(u)) \\ & + \tau\left(\frac{d}{dt}(Q_hu - P_hu)\right) \\ & + u_h - P_hu . \end{aligned}$$

Taking the inner product in H of (4.25) by $\frac{d}{dt}(Z_h - P_hZ)$, we obtain:

$$(4.26) \qquad \begin{aligned} \left|\left|\frac{d}{dt}(Z_h-P_hZ)\right|\right|_H + a\left(Z_h-P_hZ, \frac{d}{dt}(Z_h-P_hZ)\right) \leq {} & L||Z_h-Z||_V\left|\left|\frac{d}{dt}(Z_h-P_hZ)\right|\right|_H \\ & + \left|\left|\frac{d}{dt}(Z-P_hZ)\right|\right|_H\left|\left|\frac{d}{dt}(Z_h-P_hZ)\right|\right|_H \\ & + ||u_h - P_hu||_H\left|\left|\frac{d}{dt}(Z_h-P_hZ)\right|\right|_H \end{aligned}$$

Using the relation (4.12) (where $u_h - u_{oh}$ is replaced by $Z_h - P_hZ$) and the hypothesis (4.2) as well as the inequality $ab \leq \frac{1}{2\epsilon} a^2 + \frac{\epsilon}{2} b^2$, we derive from (4.26):

$$\text{(4.27)} \qquad \frac{d}{dt} a(Z_h - P_hZ, Z_h - P_hZ) \leq (L^2 + C_1^2)\|Z_h - P_hZ\|_V^2 + L^2\|Z - P_hZ\|_V^2 + \|\frac{d}{dt}(Z - P_hZ)\|_H^2 + \|u_h - P_hu\|_H^2 .$$

If we integrate (4.27) from 0 to t and then apply Gronwall's inequality, we get:

$$\text{(4.28)} \qquad \|Z_h(t) - P_hZ(t)\|_V^2 \leq \epsilon^{(1/\gamma)(L^2+C_1^2)t}\Big[\int_0^t \Big\{\frac{L^2}{\gamma}\|Z(\sigma) - P_hZ(\sigma)\|_V^2 + \frac{1}{\gamma}\|u_h(\sigma) - P_hu(\sigma)\|_H^2 + \frac{1}{\gamma}\|\frac{d}{dt}(Z(\sigma) - P_hZ(\sigma))\|_H^2\Big\}d\sigma\Big].$$

Thanks to the hypothesis (4.6) and to the estimates (4.23) and (4.24), the inequality (4.28) implies, for $0 \leq t \leq t_1$,

$$\text{(4.29)} \qquad \|Z_h(t) - P_hZ(t)\|_V^2 \leq C_2(N_2,\delta_0)e^{\tilde{c}t_1}h^{2m}\Big\{(1 + t_1)^2 + \int_0^t \|\frac{dZ}{dt}(\sigma)\|_V^2 d\sigma\Big\},$$

where $C_2(N_2,\delta_0)$ is a positive constant depending on N_2 and δ_0 only and $\tilde{c}$ is a positive constant. But, using [Henry, page 71], one easily proves that there are two constants $K_0 > 0$ and $K_1(N_2,\delta_0) > 0$ such that, for $0 < \tau \leq t$,

$$\text{(4.30)} \qquad \|\frac{du}{dt}(\tau)\|_V \leq K_1(N_2,\delta_0)e^{k_0 t_1}\frac{1}{\tau} .$$

Since $\tau Au = \tau f(u) - \tau \frac{du}{dt}$, we infer from (4.30), for $0 < \tau \leq t$,

$$\text{(4.31)} \qquad \|\tau Au\|_H \leq t_1 \sup_{v \in N_3} \|f(v)\|_H + K_1(N_2,\delta_0)e^{k_0 t_1} .$$

Finally the estimates (4.24), (4.29), (4.30) and (4.31) together with the hypothesis

(4.6)(i) allow us to write:

$$||Z(t) - Z_h(t)||_V \leq K_2(N_2,\delta_0)e^{k_3 t_1} h^m,$$

and also, for $t_0 \leq t \leq t_1$,

$$(4.32) \qquad ||u(t) - u_h(t)||_V \leq K_2(N_2,\delta_0)\frac{e^{k_3 t_1}}{t_0} h^m .$$

Remark 4.3: We also could have used the methods of [Fujita and Mizutani] for estimating $||Z_h(s) - P_h Z(s)||_V$. For the estimate of $||u(t) - u_h(t)||_V$ when u is more regular, we refer the reader to [Thomée and Wahlbin] and to [Thomée].

Remark 4.4. Let Ω be a regular or convex, bounded domain in $\mathbf{R}^n$, $n = 1,2,3$, and let $f : \mathbf{R} \to \mathbf{R}$ be a locally Lipschitz continuous function. Then, if $n = 1$, the mapping $f : u \in H^1(\Omega) \to f(u(x)) \in L^2(\Omega)$ is also locally Lipschitz continuous. If, in the cases $n = 2$ or 3, f satisfies the additional condition

$$(4.33) \qquad \forall v, \forall w \in \mathbf{R}, \ |f(v) - f(w)| \leq C(1 + |v| + |w|)^{\sigma}|v - w|$$

where

$$\sigma \leq \frac{2}{n-2} \quad \text{for} \quad n \geq 3, \quad \sigma \text{ arbitrary for } n = 2,$$

then, the mapping $f : u \in H^1(\Omega) \to f(u) \in L^2(\Omega)$ is also locally Lipschitz continuous. If the condition (4.33) is not satisfied, we have in general to work in another space than $H^1(\Omega)$ (see Section 4.2 and Example 4.2 below).

4.2. An extension of the previous result.

Let us again consider the operator A, introduced in the section 4.1, that satisfies the properties (4.1), (4.2), (4.4). Now we assume that $f : V \to H$ is no longer Lipschitz continuous. But instead, we suppose that A is a sectorial operator on a Banach space $Y \subset H$ and that $f : Y^{\alpha} \to Y$ is locally Lipschitz continuous, for a real number α, $\frac{1}{2} \leq \alpha < 1$. Furthermore, we assume that the following continuous inclusions hold:

$$(4.34) \qquad D_Y(A) \hookrightarrow Y^\alpha \hookrightarrow V \hookrightarrow Y \hookrightarrow H$$

where $D_Y(A) = \{y \in Y : Ay \in Y\}$ and $Y^\alpha = D_Y(A^\alpha)$.

We assume that all the solutions $u(t,u_0)$ of (4.3) are defined and belong to Y^α for $t \geq 0$, if $u_0 \in Y^\alpha$. Thus, the map $T_Y(t) : Y^\alpha \to Y^\alpha$, $t \geq 0$, defined by $T_Y(t)u_0 = u(t,u_0)$, becomes a C^0-semigroup on Y^α. Finally we suppose that $T_Y(t)$ admits a compact attractor $\mathcal{A}$ which attracts a bounded open set $\mathcal{O} \supset \mathcal{A}$. Then there exists an open neighborhood $\tilde{N}_1$ of $\mathcal{A}$ such that $\tilde{N}_1 \subset \mathcal{O}$ and $T_Y(t)\tilde{N}_1 \subset \tilde{N}_1$, for $t \geq 0$.

Now we introduce a function $\tilde{f}$ which is globally Lipschitz continuous from V into H, coincides with f on $\mathcal{O}$, and we consider the equation

$$(4.35) \qquad \begin{cases} \dfrac{d\tilde{u}}{dt} + A\tilde{u} = \tilde{f}(\tilde{u}), \\ \\ \tilde{u}(0) = u_0 \, . \end{cases}$$

Obviously, if $u_0 \in \tilde{N}_1$, $\tilde{u}(t,u_0) = u(t,u_0)$ for $t \geq 0$. Let $(V_h)_h$ be the family of finite-dimensional subspaces of V introduced in Section 4.1. We suppose that the spaces V_h are included in Y^α, satisfy the conditions (4.6) and the two following assumptions

(4.36)(i) for any β, $\alpha < \beta \leq 1$, there exists a constant $\theta(\alpha,\beta) > 0$, such that, for v in Y^β,

$$\|v - P_h v\|_{Y^\alpha} \leq Ch^{2m\theta(\alpha,\beta)}\|v\|_{Y^\beta} \, ,$$

and

(4.36)(ii) there exists a constant θ_α, $0 < \theta_\alpha < \frac{1}{2}$, such that, for any v_h in V_h,

$$||v_h||_{Y^\alpha} \leq Ch^{-2m\theta_\alpha}||v_h||_V .$$

We consider the approximate problem

$$(4.37)_h \quad \begin{cases} \dfrac{d\tilde{u}_h}{dt} + A_h\tilde{u}_h = Q_h\tilde{f}(\tilde{u}_h) \\ \tilde{u}_h(0) = u_{oh} , \end{cases}$$

for $u_{oh} \in V_h$. We introduce the map $\tilde{T}_h(t) : V_h \to V_h$ given by $\tilde{T}_h(t)u_{oh} = \tilde{u}_h(t,u_{oh})$. Since $\tilde{f}$ is globally Lipschitz continuous, $\tilde{T}_h(t)u_{oh}$ exists for any $t \geq 0$.

Theorem 4.2. *Under the above hypotheses, there exists* $h_0 > 0$ *such that, for* $h \leq h_0$, $\tilde{T}_h(t)$ *admits a compact attractor* A_h *which attracts the open set* $\tilde{N}_1 \cap V_h$ *(where* $\tilde{N}_1$ *is given above). Moreover,* $\delta_{Y^\alpha}(\tilde{A}_h,A) \to 0$ *as* $h \to 0$.

Proof. Let $t_0 > 0$ be a fixed real number. For any $t_1 > t_0$, we are going to estimate $||T_Y(t)u_{oh} - \tilde{T}_h(t)u_{oh}||_{Y^\alpha}$ for $t_0 \leq t \leq t_1$, when $u_{oh} \in \tilde{N}_1$. We set $u(t) = T_Y(t)u_{oh}$, $\tilde{u}_h(t) = \tilde{T}_h(t)u_{oh}$. Recall that $u(t) = \tilde{u}(t)$. Due to the conditions (4.36), we have:

$$\begin{aligned} ||u(t) - \tilde{u}_h(t)||_{Y^\alpha} &\leq ||\tilde{u}(t) - P_h\tilde{u}(t)||_{Y^\alpha} + ||P_h\tilde{u}(t) - \tilde{u}_h(t)||_{Y^\alpha} \\ &\leq Ch^{2m\theta(\alpha,\beta)}||u(t)||_{Y^\beta} + Ch^{-2m\theta_\alpha}||P_h\tilde{u}(t) - \tilde{u}_h(t)||_V \end{aligned}$$

where $\alpha < \beta < 1$.

Arguing as in Section 4.1 (see Estimate (4.32)), we get:

$$h^{-2m\theta_\alpha}||P_h\tilde{u}(t) - \tilde{u}_h(t)||_V \leq K_2(\tilde{N}_1)\,\frac{e^{k_3 t_1}}{t_0}\,h^{2m(1/2-\theta_\alpha)} .$$

Finally, by using [Henry, page 57], we deduce from the above estimates, for $t_0 \leq t \leq t_1$,

$$(4.38) \qquad \|u(t) - \tilde{u}_h(t)\|_{Y^\alpha} \leq K_4(\beta,\tilde{N}_1)\, \frac{e^{k_5 t_1}}{t_0}\, \sup(h^{2m(1/2-\theta_\alpha)} h^{2m\theta(\alpha,\beta)})$$

Since $\theta_\alpha < \frac{1}{2}$, (4.38) implies that the hypotheses of Proposition 1.1 hold and Theorem 4.2 is proven.

Example 4.2. Consider the equation

$$(4.39) \qquad \begin{cases} \dfrac{du}{dt} - \Delta u = f(u), \\ u/\partial\Omega = 0, \\ u(t)_{/t=0} = u_0 \end{cases}$$

where, for instance, Ω is a convex polygonal domain in $\mathbf{R}^2$. If the function $f : R \to R$ is locally Lipschitz continuous, but does not satisfy the condition (4.33), we cannot work in the space $V = H_0^1(\Omega)$. The map $f : w \in Y^\alpha \to f(w) \in Y$ is locally Lipschitz continuous if $Y = L^2(\Omega)$ and $\alpha > \frac{1}{2}$, or, if $Y = L^p(\Omega)$, $p > 2$ and $\alpha \geq \frac{1}{2}$. (Indeed in both cases, $Y^\alpha \hookrightarrow L^\infty(\Omega)$).

Now assume that (4.39) admits an attractor $\mathcal{A}$ in Y^α which attracts a bounded set $\mathcal{O} \supset \mathcal{A}$. So we can introduce the quantity

$$(4.39) \qquad B_0 = \max_{v\ \mathcal{O}} \|v\|_{L^\infty(\Omega)}\,.$$

One easily constructs a function $\tilde{f}$ satisfying

$$(4.40) \qquad \tilde{f}(x) = \begin{cases} f(x) & \text{for} \quad |x| \leq B_1, \\ 0 & \text{for} \quad |x| \geq 2B_1 \end{cases}$$

The map $\tilde{f} : w \in V \to \tilde{f}(w) \in H$ is globally Lipschitz continuous and coincides with f on $\mathcal{O}$.

Let us give an example of spaces V_h in the case $Y^{\alpha} = H^{2\alpha}(\Omega) \cap H_0^1(\Omega)$, $\frac{1}{2} < \alpha < 1$. Let $(\mathcal{T}_h)_h$ be a uniformly regular family of triangulations in the sense of [Ciarlet]. We set:

$$V_h = \{v_h \in C^1(\bar{\Omega}) \cap H_0^1(\Omega) : v_h|_K \in P_3(K),\ \forall K \in \mathcal{T}_h\}$$

where $P_3(K)$ is the space of all polynomials of degree $\leqslant 3$ on K. Then, of course, the hypotheses (4.6) are satisfied with $m = 1$. Condition (4.36)(i) and (4.36)(ii) hold with $\theta(\alpha,\beta) = \beta - \alpha$ and $\theta_{\alpha} = \alpha - \frac{1}{2}$.

5. Semi-discretization in Time of Some Parabolic Problems.

We keep the same notations and the same assumptions as in Section 4.1, but here we moreover assume that the operator A is self-adjoint and has a compact resolvent. (The generalization of the following results to the case where A is not self-adjoint, but satisfies the conditions (4.2) and (4.4) is left to the reader). As in Section 4.1, we assume that $\gamma_0 = 0$ and we consider the nonlinear equation:

$$(5.1) \qquad \begin{cases} \dfrac{du}{dt} + Au = f(u), \\ \\ u(0) = u^0, \end{cases}$$

where $u^0 \in V$ and $f \in C^2(V;H)$, for instance. The hypotheses on f can be weakened. Now let us turn to a semi-discretization in time of Equation (5.1) by a single step method. More precisely, let k be a positive time increment, let $t_n = nk$, $n \geqslant 0$, and define an approximation u_n of the solution u of (5.1) at the time t_n by the recursion formula

$$(5.1)_k \qquad \begin{cases} u_{n+1} = (1 - (1-\theta)kA)(1 + \theta kA)^{-1}u_n + k(1 + \theta kA)^{-1}f(u_n) \\ \\ u_0 = u^0, \end{cases}$$

where $\frac{1}{2} < \theta \leqslant 1$.

Remark 5.1. The results that we are going to prove below are also valid if we replace $f(u_n)$ in $(5.1)_k$ by $f(\theta u_{n+1} + (1-\theta)u_n)$. But then the "linearized" scheme $(5.1)_k$ becomes a nonlinear one.

More generally the following results are also true if we replace $(5.1)_k$ by a scheme that is strictly accurate of order 1 in the sense of [Brenner, Crouzeix, Thomée] and is of the form:

$$(5.2)_k \qquad \begin{cases} u_{n+1} = r(kA)u_n + k \sum_{j=1}^{m} q_j(kA)f(u_n), \\ u_0 = u^0, \end{cases}$$

where r, $q_1,\dots,q_m$ are rational functions of the variable z which are bounded, as well as $zq_j(z)$, $1 \leq j \leq m$, for $z \geq 0$, and where $|r(z)| < 1$, for $z \geq 0$, and $|r(\infty)| \neq 1$. The proof, in the case of the schemes $(5.2)_k$, uses the same arguments as below and the property that r(z) can be written as $\frac{1 - zs(z)}{1 + \sigma z}$ where σ is an adequate positive constant (for more details, see [Raugel]).

Now we introduce the mapping $T_k \in L(V,V)$ defined by $T_k u^0 = u_1$ where u_1 is given by the formula $(5.1)_k$. For any integer $n \geq 1$, $T_k^n u^0 = u_n$. Let us remark that T_k is well defined on the whole space V and that $T_k^n : \mathbf{N} \to C^0(V,V)$ is a discrete semigroup. Although the sections 1 and 2 deal with C^0-semigroups $T(t) : \mathbf{R}^+ \to C^0(V;V)$ only, the definitions and the results contained there obviously extend to the discrete semigroups. For instance, a set $B \subset V$ is said to attract a set $C \subset V$ under T_k if, for any $\epsilon > 0$, there is an integer $n_0 = n_0(B,C,\epsilon)$ such that $T_k^N \subset \mathcal{N}(B,\epsilon)$ for $n \geq n_0$ (the definitions of a local attractor and an attractor are unchanged; for more details, see [Hale, 1], for instance).

Here we suppose that the map $T(t) : V \to V$, $t \geq 0$, defined by $T(t)u^0 = u(t)$ where u(t) is the solution of (5.1), admits a local compact attractor $\mathcal{A}$ which attracts a bounded open set $\mathcal{O}$, $\mathcal{O} \supset \mathcal{A}$.

Theorem 5.1. *Under the above hypotheses, there exists* $k_0 > 0$, *such that, for* $k \leq k_0$, T_k^n *admits a local compact attractor* $\mathcal{A}_k$, *which attracts an open set* N_1 *where* N_1 *is independent of* k, $N_1 \supset \mathcal{A}_k$ *for every* k. *Moreover* $\delta_V(\mathcal{A}_k,\mathcal{A}) \to 0$ *as* $k \to 0$.

The remainder of this section will be devoted to the proof of Theorem 5.1. But, beforehand, let us recall the following discrete analogue of Gronwall's lemma, the proof of which is left to the reader.

Lemma 5.2. *Let* $(a_n)_n$, $(b_n)_{n'}$ $(c_n)_n$ *be three sequences of positive real numbers such that* $(c_n)_n$ *is monotonically increasing and*

$$a_n + b_n \leqslant c_n + \lambda \sum_{m=0}^{n-1} a_m \quad \textit{for} \quad n \geqslant 1 \textit{ and } \lambda > 0, \tag{5.3}$$

with

$$a_0 + b_0 \leqslant c_0 .$$

Then, these sequences also satisfy

$$a_n + b_n \leqslant c_n \exp(\lambda n) \quad \textit{for} \quad n \geqslant 0. \tag{5.4}$$

Only for the sake of simplicity, we consider that the space V is equipped with the norm:

$$\forall v \in V, \ ||v||_V = (Av,v)^{1/2}. \tag{5.5}$$

Hence the dual norm on V' is given by

$$\forall v' \in V', \quad ||v'||_{V'} = (A^{-1}v',v')^{1/2} .$$

Proof of Theorem 5.1. In order to prove Theorem 5.1 we shall apply the following modified version of Theorem 2.4, the proof of which is left to the reader. Clearly the conclusions of Theorem 2.4 and hence of Theorem 5.1 hold, if the following conditions are satisfied:

There exist four positive constants k_0, δ_0, δ_1, α_0, with $\alpha_0 > k_0$, and two open neighborhoods N_1, N_2, of A, with $N_1 \subset N_2$, such that, for $0 < k \leqslant k_0$

(i) T_k is an asymptotically smooth map (this condition holds in particular, if $T_k = T_{1k} + T_{2k}$, where T_{1k} is completely continuous and t_{2k} is a linear strict contraction);

(ii) $T(t)\mathcal{N}_1 \subset \mathcal{N}_2$ for $t \geq 0$,

(iii) $T_k^n\mathcal{N}_1 \subset \mathcal{N}_2$ for $0 \leq n \leq \frac{\alpha_0}{k}$,

(iv) $T_k\mathcal{N}(\mathcal{N}_2,\delta_0) \subset \mathcal{N}_3$ where $\mathcal{N}_3 = \mathcal{N}(\mathcal{N}_2,\delta_0 + \delta_1)$;

and

(v) for any $\alpha_1 > \alpha_0$, there exists a constant $k_0(\alpha_1,\mathcal{N}_3)$ with $0 < k_0(\alpha_1,\mathcal{N}_3) < k_0$, and a function $\eta(k,\alpha_1,\mathcal{N}_3)$ defined for $0 < k \leq k_0(\alpha,\mathcal{N}_3)$ such that

$$\lim_{k\to\infty} \eta(k,\alpha_1,\mathcal{N}_3) = 0, \tag{5.6}$$

and, for any $0 < k \leq k_0(\alpha_1,\mathcal{N}_3)$, if $u^0 \in \mathcal{N}_3$ has the property that $T_n^k u^0$ and $T(nk)u^0$ belong to $\mathcal{N}_3$ for $0 \leq n \leq \frac{\alpha_2}{k}$ and $0 \leq nk \leq \alpha_2 + k_0$ respectively (where $\alpha_0 \leq \alpha_2 \leq \alpha_1$), then

$$\|T_n^k u^0 - T(nk)u^0\|_V \leq \eta(k,\alpha_1,\mathcal{N}_3) \text{ for } \frac{\alpha_0}{k} \leq n \leq \frac{\alpha_2}{k}. \tag{5.7}$$

Now let us show in four steps that the above conditions are satisfied.

1) By $(5.1)_k$, we can write, for any $u^0 \in V$,

$$T_k u^0 = [(1+\theta kA)^{-1}u^0 + k(1+\theta kA)^{-1}f(u^0)] - (1-\theta)kA(1+\theta kA)^{-1}u^0$$
$$\equiv T_{1k}u^0 + T_{2k}u^0.$$

Let B be a bounded set in H; for any $v \in B$, we have $|\,|kA(1+\theta kA)^{-1}v|\,|_H \leq \|v\|_H$. Hence, for any fixed positive k, $(1+\theta kA)^{-1}B$ is a bounded set in D(A). Since $D(A) \subset V$ is a compact embedding, this proves that T_{1k} is completely continuous. On the other hand, as A is an elliptic operator, T_{2k}, for $k > 0$, is a linear strict contraction as soon as $2\theta - 1 > 0$. Condition (i) is proved.

2) As $\mathcal{A}$ is a compact attractor, there is a bounded open neighborhood $\mathcal{N}_1$ of $\mathcal{A}$ such that $\mathcal{N}_1 \subset \mathcal{O}$ and $T(t)\mathcal{N}_1 \subset \mathcal{N}_1$, for $t \geq 0$. Let $B_0 = \max_{v\in\mathcal{N}_1} |\,|v|\,|_V$ and $B_1 = \max_{v\in\mathcal{N}_1} |\,|f(v)|\,|_H$; we set $\epsilon_0 = 4(B_0^2 + B_1^2)^{1/2}$ and $\mathcal{N}_2 = \mathcal{N}(\mathcal{N}_1,\epsilon_0)$. Finally, we

choose a real number $\delta_0 > 0$ and we set $\delta_1 = 2[(B_0 + \epsilon_0 + \delta_0)^2 + B_2^2]^{1/2}$ where $B_2 = \max_{v \in \mathcal{N}(\mathcal{N}_2,\delta_0)} ||f(v)||_H$.

Let us remark that the condition (iii) is an immediate consequence of the following property:

$$\text{(A)} \quad \begin{cases} \text{there exists a constant } \alpha_0 > 0 \text{ independent of k, such that, for} \\ \text{any } u^0 \in \mathcal{N}_1, \text{ if } T_k^n u^0 \text{ belongs to } \mathcal{N}(u^0,\epsilon_0), \text{ for } 0 \leq n \leq \beta(k,u^0)/k, \\ \text{with } 0 \leq \beta(k,u^0) \leq \alpha_0 - k, \text{ then } T_k^n u^0 \text{ belongs to } \mathcal{N}(u^0,\epsilon_0) \text{ for} \\ 0 \leq n \leq \beta(k,u^0)/k + 1. \end{cases}$$

Let $u^0 \in \mathcal{N}_1$. We set $u_n = T_k^n u^0$, $\tilde{u}_n = u_n - u^0$ and we assume that, for $0 \leq n \leq \beta(k,u^0)/k$, $T_k^n u^0 \in \mathcal{N}(u^0,\epsilon_0)$. By $(5.1)_k$, we have

$$\tilde{u}_n - \tilde{u}_{n-1} + kA(\theta\tilde{u}_n + (1-\theta)\tilde{u}_{n-1}) = kf(u_{n-1}) - kAu^0. \tag{5.8}$$

Taking the inner product in H of (5.8) by $\tilde{u}_n - \tilde{u}_{n-1}$, we obtain

$$\begin{aligned} ||\tilde{u}_n - \tilde{u}_{n-1}||_H^2 + \frac{k}{2}||\tilde{u}_n||_V^2 - \frac{k}{2}||\tilde{u}_{n-1}||_V^2 + \frac{k}{2}(2\theta-1)||\tilde{u}_n - \tilde{u}_{n-1}||_V^2 \\ \leq k(f(u_{n-1}) - f(u^0),\tilde{u}_n - \tilde{u}_{n-1}) \\ + k(f(u^0),\tilde{u}_n - \tilde{u}_{n-1}) + k(Au^0,\tilde{u}_n - \tilde{u}_{n-1}), \end{aligned}$$

or also,

$$||\tilde{u}_n||_V^2 - ||\tilde{u}_{n-1}||_V^2 \leq k\, L^2||\tilde{u}_{n-1}||_V^2 + kB_1^2 + (Au^0,\tilde{u}_n - \tilde{u}_{n-1}),$$

where $L > 0$ is the Lipschitz constant of f on $\mathcal{P}_3$.

Summation over n yields:

$$||\tilde{u}_{m+1}||_V^2 \leq kL^2 \sum_{n=0}^{m} ||\tilde{u}_n||_V^2 + k(m+1)B_1^2 + ||u^0||_V||\tilde{u}_{m+1}||_V \tag{5.9}$$

where m is the integral part of $\beta(k,u^0)/k$. Using lemma 5.2, we infer from (5.9),

(5.10) $$||\tilde{u}_{m+1}||_V^2 \leq [B_0^2 + + 2k(m+1)B_1^2]\exp(2kL^2(m+1)).$$

Let now α_0 be a positive constant such that

(5.11) $$[B_0^2 + 2\alpha_0 B_1^2]\exp(2L^2\alpha_0) < \epsilon_0$$

and choose k_0 such that $0 < k_0 < \alpha_0$. Then one deduces from (5.10) that $\tilde{u}_{m+1} \in N(u^0,\epsilon_0)$ if $m+1 \leq \alpha_0/k$, for $0 < k \leq k_0$. Thus, Property (A) is shown. As the proof of the condition (iv) uses similar estimates, it is left to the reader.

3) Some auxiliary estimates.

We shall estimate $k \sum_{n=0}^{m} ||T(nk)u^0 - T_k^n u^0||_V^2$ and $\sum_{n=0}^{m} ||(T((n+1)k)u^0 - T_k^{n+1}u^0) - (T(nk)u^0 - T_k^n u^0)||_H^2$ for $0 \leq m \leq \alpha_1/k$, when $T_n^k u^0$ and $T(nk)u^0$ belong to N_s for $0 \leq n \leq m$ and $0 \leq nk \leq mk + k_0$ respectively.

We set $t_n = nk$ and $e_n \equiv T_k^n u^0 - T(nk)u^0 \equiv u_n - u(t_n)$. As it was pointed out in [Raugel, proof of Theorem 2.2], one easily shows that

(5.12) $$k \sum_{n=0}^{m} ||e_n||_V^2 + \theta k||e_{m+1}||_V^2 - k \sum_{n=0}^{m} \theta(1-\theta)||e_{n+1} - e_n||_V^2$$
$$\leq k \sum_{n=0}^{m} ||\theta e_{n+1} + (1-\theta)e_n||_V^2.$$

From the equations (5.1) and $(5.1)_k$, we infer:

(5.13) $$e_{n+1} - e_n + kA(\theta e_{n+1} + (1-\theta)e_n) = k(f(u_n) - f(u(t_n))$$
$$- \int_{t_n}^{t_{n+1}} \left(\frac{du}{dt}(s) - \frac{du}{dt}(t_n)\right)ds$$
$$+ \theta kA(u(t_n) - u(t_{n+1})).$$

Taking the inner product in H of (5.13) by $\theta e_{n+1} + (1-\theta)e_n + \gamma_1(e_{n+1} - e_n)$ where $\gamma_1 > 0$, we obtain the following inequality:

$$\frac{1}{2}\|e_{n+1}\|_H^2 - \frac{1}{2}\|e_n\|_H^2 + \frac{1}{2}(2\theta-1)\|e_{n+1} - e_n\|_H^2 + k\|\theta e_{n+1} + (1-\theta)e_n\|_V^2$$

$$+ \gamma_1\|e_{n+1} - e_n\|_H^2 + \frac{k\gamma_1}{2}\|e_{n+1}\|_V^2 - \frac{k\gamma_1}{2}\|e_n\|_V^2 + \frac{k\gamma_1}{2}(2\theta-1)\|e_{n+1} - e_n\|_V^2$$

$$\leq kL\|e_n\|_V[(\theta+\gamma_1)\|e_{n+1} - e_n\|_H + \|e_n\|_H]$$

$$+ \theta k\|u(t_n) - u(t_{n+1})\|_V[\|\theta e_{n+1} + (1-\theta)e_n\|_V + \gamma_1\|e_{n+1} - e_n\|_V]$$

$$+ \left\| \int_{t_n}^{t_{n+1}} \left(\frac{du}{dt}(s) - \frac{du}{dt}(t_n)\right)ds\right\|_{V'}[\|\theta e_{n+1} + (1-\theta)e_n\|_V + \gamma_1\|e_{n+1} - e_n\|_V].$$

Using the inequality $ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$ several times, we derive from the above estimate:

$$\|e_{n+1}\|_H^2 - \|e_n\|_H^2 + k\|\theta e_{n+1} + (1-\theta)e_n\|_V^2 + \gamma_1\|e_{n+1} - e_n\|_H^2 \tag{5.14}$$

$$+ k\gamma_1\|e_{n+1}\|_V^2 - k\gamma_1\|e_n\|_V^2 + k\gamma_1(2\theta-1)\|e_{n+1} - e_n\|_V^2$$

$$\leq \left[\frac{k^2\theta^2L^2}{2\theta-1} + \frac{\gamma_1 k^2L^2}{2} + k\epsilon_0\right]\|e_n\|_V^2 + \frac{kL^2}{\epsilon_0}\|e_n\|_H^2$$

$$+ k\left[2\theta^2 + \frac{4\gamma_1\theta^2}{2\theta-1}\right]\|u(t_n) - u(t_{n+1})\|_V^2$$

$$+ \left[2 + \frac{4\gamma_1^2}{2\theta-1}\right]\int_{t_n}^{t_{n+1}} \left\|\frac{du}{dt}(s) - \frac{du}{dt}(t_n)\right\|_{V'}^2 ds$$

where $\epsilon_0 > 0$ is a small enough constant.

Summation of (5.14) over n yields:

$$\|e_{m+1}\|_H^2 + k\sum_{n=0}^{m}\|\theta e_{n+1} + (1-\theta)e_n\|_V^2 + \gamma_1\sum_{n=0}^{m}\|e_{n+1} - e_n\|_H^2$$

$$+ k\gamma_1\|e_{m+1}\|_V^2 + \frac{k\gamma_1}{2}(2\theta-1)\sum_{n=0}^{m}\|e_{n+1} - e_n\|_V^2$$

$$\leq k\left[\frac{k\theta^2L^2}{2\theta-1} + \frac{\gamma_1 kL^2}{2} + \epsilon_0\right]\sum_{n=0}^{m} ||e_n||_V^2 + \frac{kL^2}{\epsilon_0}\sum_{n=0}^{m} ||e_n||_H^2$$

$$+ k\left[2\theta^2 + \frac{4\gamma_1\theta^2}{2\theta-1}\right]\sum_{n=0}^{m} ||u(t_n) - u(t_{n+1})||_V^2$$

$$+ \left[2 + \frac{4\gamma_1}{2\theta-1}\right]\sum_{n=0}^{m}\int_{t_n}^{t_{n+1}} ||\frac{du}{dt}(s) - \frac{du}{dt}(t_n)||_{V'}^2\, ds$$

Now we set $\gamma_2 = \sup(1,\frac{2\theta(1-\theta)}{2\theta-1})$ and we choose $k_0 > 0$ and $\epsilon_0 > 0$ such that, for $0 < k \leq k_0$, $\frac{k\theta^2L^2}{2\theta-1} + \frac{\gamma_1 kL^2}{2} + \epsilon_0 < \frac{1}{2}$. Then, thanks to (5.12), we deduce from the previous inequality that

$$||e_{m+1}||_H^2 + \frac{k}{2}\sum_{n=0}^{m} ||e_n||_V^2 + \sum_{n=0}^{m} ||e_{n+1} - e_n||_H^2 \tag{5.15}$$

$$\leq \frac{kL^2}{\epsilon_0}\sum_{n=0}^{m} ||e_n||_H^2 + C(\theta)k\sum_{n=0}^{m} ||u(t_{n+1}) - u(t_n)||_V^2$$

$$+ c(\theta)\sum_{n=0}^{m}\int_{t_n}^{t_{n+1}} ||\frac{du}{dt}(s) - \frac{du}{dt}(t_n)||_{V'}^2\, ds$$

Due to lemma 5.2 we infer from (5.15):

$$||e_{m+1}||_H^2 + \frac{k}{2}\sum_{n=0}^{m} ||e_n||_V^2 + \sum_{n=0}^{m} ||e_{n+1} - e_n||_H^2 \leq c(\theta)\exp\left[\frac{kL^2(m+1)}{\epsilon_0}\right] \times \tag{5.16}$$

$$\left[k\sum_{n=0}^{m} ||u(t_{n+1}) - u(t_n)||_V^2 + \sum_{n=0}^{m}\int_{t_n}^{t_{n+1}} ||\frac{du}{dt}(s) - \frac{du}{dt}(t_n)||_{V'}^2\, ds\right]$$

Let us set: $B_3 = \max_{v\in N_3} ||v||_V$. Then we have:

$$\sum_{n=0}^{m} ||u(t_{n+1}) - u(t_n)||_N^2 \leq 2B_3^2 + k\int_{t_1}^{t_{m+1}} ||\frac{du}{dt}||_V^2\, ds \tag{5.17}$$

By [Henry, page 71], there exist two constants $K_0 > 0$ and $K_1(N_3) > 0$ such that, for $0 \leqslant K \leqslant mk + k_0$,

$$(5.18)\qquad t\,\left|\left|\frac{du}{dt}\right|\right|_V + t^{1/2}\,\left|\left|\frac{du}{dt}\right|\right|_H \leqslant K_1(N_3)\, e^{K_0(\alpha_1+k_0)}\,.$$

From (5.17) and (5.18), we derive:

$$(5.19)\qquad k\sum_{n=0}^{m} ||u(t_{n+1}) - u(t_n)||_V^2 \leqslant k(2B_3^2 + K_1^2(N_3)e^{2K_0(\alpha_1+k_0)}).$$

On the other hand, we have

$$(5.20)\qquad \sum_{n=0}^{m}\int_{t_n}^{t_{n+1}} \left|\left|\frac{du}{dt}(s) - \frac{du}{dt}(t_n)\right|\right|_{V'}^2 ds \leqslant 2k \sup_{t\in[0,t]} \left|\left|\frac{du}{dt}\right|\right|_{V'}^2$$

$$+\, k^2\int_{t_1}^{t_{m+1}} \left|\left|\frac{d^2u}{dt^2}\right|\right|_{V'}^2 ds.$$

Since $A^{-1/2}\frac{du}{dt} = A^{-1/2} f(u) - A^{1/2}u$ and $\left|\left|\frac{du}{dt}\right|\right|_{V'}^2 = \left|\left|A^{-1/2}\frac{du}{dt}\right|\right|_H^2$, we obtain:

$$(5.21)\qquad \sup_{0\leqslant t\leqslant t_{m+1}} \left|\left|\frac{du}{dt}\right|\right|_{V'}^2 \leqslant B_3^2 + \max_{v\in N_3} ||f(v)||_H^2.$$

Since the inequalities (4.22) and (5.18) hold, $f'(u)t\frac{du}{dt} + \frac{du}{dt}$ belongs to $L^2([0,t_{m+1}];H)$ and one easily proves that the function $t\frac{du}{dt}$ satisfies the equation:

$$(5.22)\qquad \begin{cases} \text{(i)}\ \left(\frac{d}{dt}\left(t\,\frac{du}{dt}\right),\phi\right) + a\left(t\frac{du}{dt},\phi\right) = \left(f'(u)t\,\frac{du}{dt},\phi\right) + \left(\frac{du}{dt},\phi\right) \text{ for } \phi\in V; \\[2ex] \text{(ii)}\ \left(t\frac{du}{dt}\right)_{/t=0} = 0. \end{cases}$$

(Hence $t\,\frac{du}{dt}$ belongs to the space $H^1([0,t_{m+1}];H)$.

For $t > 0$, equation (5.22)(i) also can be written as

$$\left(t\frac{d^2u}{dt^2},\phi\right) + a\left(t\frac{du}{dt},\phi\right) = \left(f'(u)t\frac{du}{dt},\phi\right) \text{ for any } \phi \in V. \tag{5.23}$$

Let us set $\phi = \frac{du}{dt}$ in (5.23); then, after an integration from 0 to t_{m+1}, we obtain:

$$\frac{1}{2}\left[t\left|\left|\frac{du}{dt}\right|\right|_H^2\right](t_{m+1}) + \gamma\int_0^{t_{m+1}} t\left|\left|\frac{du}{dt}\right|\right|_V^2 dt \leq \frac{1}{2}\int_0^{t_{m+1}}\left|\left|f'(u)t\frac{du}{dt}\right|\right|_H^2 dt$$
$$+ \frac{3}{2}\int_0^{t_{m+1}}\left|\left|\frac{du}{dt}\right|\right|_H^2 dt + \left[\frac{1}{2}t\left|\left|\frac{du}{dt}\right|\right|_H^2\right](0). \tag{5.24}$$

Since

$$\int_0^{t_{m+1}}\left|\left|f'(u)t\frac{du}{dt}\right|\right|_H^2 dt \leq \sup_{u\in\mathcal{N}_3}||f'(u)||^2_{\mathcal{L}(V;H)}\int_0^{t_{m+1}} t^2\left|\left|\frac{du}{dt}\right|\right|_V^2 dt,$$

we deduce from (5.24), by using (4.22) and (5.18), that

$$\int_0^{t_{m+1}} t\left|\left|\frac{du}{dt}\right|\right|_V^2 dt \leq K_2(\mathcal{N}_3)\, e^{K_3(\alpha_1+k_0)}, \tag{5.25}$$

$K_2(\mathcal{N}_3)$ and K_3 are two positive constants.

Now let us set $\phi = A^{-1}\frac{d^2u}{dt^2}$ in (5.23); we get

$$\frac{1}{2}\int_0^{t_{m+1}} t\left|\left|A^{-1/2}\frac{d^2u}{dt^2}\right|\right|_H^2 dt + \frac{1}{2}\left[t\left|\left|\frac{du}{dt}\right|\right|_H^2\right](t_{m+1})$$
$$\leq \frac{1}{2}\int_0^{t_{m+1}}\left|\left|A^{-1/2}f'(u)t^{1/2}\frac{du}{dt}\right|\right|_H^2 dt + \int_0^{t_{m+1}}\left|\left|\frac{du}{dt}\right|\right|_H^2 dt$$
$$+ \frac{1}{2}\left[t\left|\left|\frac{du}{dt}\right|\right|_H^2\right](0),$$

which implies, thanks to (4.22), (5.18) and (5.25),

$$\int_0^{t_{m+1}} \left|\left|A^{-1/2} \frac{d^2u}{dt^2}\right|\right|_H^2 dt \leq K_4(\mathcal{N}_3)e^{K_5(\alpha_1+k_0)}, \tag{5.26}$$

where $K_4(\mathcal{N}_3)$ and K_5 are two positive constants.

From (5.20), (5.21) and (5.26), we derive:

$$\sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} \left|\left|\frac{du}{dt}(s) - \frac{du}{dt}(t_n)\right|\right|_{V'}^2 ds \leq k\Big[2(B_3^2 + \max_{v\in\mathcal{N}_3} ||f(v)||_H^2 + K_4^2(\mathcal{N}_3)e^{2K_5(\alpha_1+k_0)}\Big] \tag{5.27}$$

Finally, from (5.16), (5.19) and (5.27), we infer:

$$||e_{m+1}||_H^2 + \frac{k}{2}\sum_{n=0}^{m} ||e_n||_V^2 + \sum_{n=0}^{m} ||e_{n+1} - e_n||_H^2 \leq k\, K_6(\mathcal{N}_3)\, e^{K_7(\alpha_1+k_0)}, \tag{5.28}$$

where $K_6(\mathcal{N}_3)$ and K_7 are two positive constants.

4) Estimate of $||T(nk)u^0 - T_n^k u^0||_V$ for $\alpha_0/k \leq n \leq m+1$, when $T_n^k u^0$ and $T(nk)u^0$ belong to $\mathcal{N}_3$ for $0 \leq n \leq m$ and $0 \leq nk \leq mk + k_0$ respectively, where $\alpha_0/k < m \leq \alpha_1/k$.

To this end, we at first estimate the term $||t_n(T(t_n)u^0 - T_n^k u^0)||_V$ for $0 \leq n \leq m$. Formula $(5.1)_k$ gives:

$$t_{n+1}u_{n+1} - t_n u_n + kA(\theta t_{n+1}u_{n+1} + (1-\theta)t_n u_n) = kt_n f(u_n) + ku_{n+1} + \theta k^2 Au_{n+1}. \tag{5.29}$$

Let us set: $\bar{e}_n = t_n(u_n - u(t_n))$. From (5.29) and from the equation (5.1) we deduce:

$$(5.30)\qquad \bar{e}_{n=1} - \bar{e}_n + kA(\theta\bar{e}_{n+1} + (1-\theta)\bar{e}_n)$$

$$= kt_n(f(u_n) - f(u(t_n)))$$

$$- \int_{t_n}^{t_{n+1}} \left[\frac{d}{ds}(su(s)) - \frac{d}{ds}\,(su(s)_{/s=t_n}\right] ds$$

$$+ \theta kA(t_n u(t_n) - t_{n+1}u(t_{n+1}))$$

$$+ ke_n + k(u_{n+1} - u_n) + \theta k^2 Au_{n+1}.$$

Taking the inner product in H of (5.30) by $\bar{e}_{n+1} - \bar{e}_n$, we obtain:

$$||\bar{e}_{n+1} - \bar{e}_n||_H^2 + \frac{k}{2}||\bar{e}_{n+1}||_V^2 - \frac{k}{2}||\bar{e}_n||_V^2 + \frac{k}{2}(2\theta-1)||\bar{e}_{n+1} - \bar{e}_n||_V^2$$

$$\leq k||\bar{e}_{n+1} - \bar{e}_n||_H\,[L||\bar{e}_n||_V + ||e_n||_H + ||u_{n+1} - u_n||_H]$$

$$+ k||\bar{e}_{n+1} - \bar{e}_n||_V[\theta k||u_{n+1}||_V + \theta||t_n u(t_n) - t_{n+1}u(t_{n+1})||$$

$$+ \frac{1}{k}\,\left|\left|\int_{t_n}^{t_{n+1}}\left[\frac{d}{ds}(su(s)) - \frac{d}{ds}(su(s))_{/s=t_n}\right]ds\right]\right.$$

or also

$$||\bar{e}_{n+1}||_V^2 - ||\bar{e}_n||_V^2 \leq 2k\ L^2||\bar{e}_n||_V^2 + 2k||e_n||_H^2 + 2k||u_{n+1} - u_n||_H^2$$

$$+ \frac{3}{2\theta-1}\left[k^2||u_{n+1}||_V^2 + k\int_{t_n}^{t_{n+1}}||\frac{d}{ds}(su(s))||_V^2\,ds\right.$$

$$\left. + k\int_{t_n}^{t_{n+1}}||\frac{d^2}{ds^2}(su(s))||^2_{V'}\,ds\right].$$

Summing the previous inequality over n and applying lemma 5.2, we get:

$$(5.31)\qquad \max_{0\le n\le m+1} \|\bar{e}_{n+1}\|_V^2 \le C \exp kL^2(m+1)\Big[k \sum_{n=0}^{m} \|e_n\|_H^2$$

$$+ k \sum_{n=0}^{m} \|u_{n+1} - u_n\|_H^2 + k^2 \sum_{n=0}^{m} \|u_{n+1}\|_V^2$$

$$+ k \int_0^{t_{m+1}} \|\frac{d}{ds}(su(s))\|_V^2\, ds + k \int_0^{t_{m+1}} \|\frac{d^2}{ds^2}su(s)\|_{V'}^2\, ds\Big]$$

But

$$(5.32)\qquad k \sum_{n=0}^{m} \|u_{n+1} - u_n\|_H^2 \le 2k \sum_{n=0}^{m} \|e_{n+1} - e_n\|_H^2 + k^2 \int_0^{t_{m+1}} \|\frac{du}{dt}\|_H^2\, ds$$

and

$$(5.33)\qquad k^2 \sum_{n=0}^{m} \|u_{n+1}\|_V^2 \le k^2(m+1)B_3^2 .$$

Finally we derive from (5.31), (5.32), (5.33), (5.28), as well as from (4.22), (5.18) and (5.26) that

$$\max_{0\le n\le m+1} \|\bar{e}_{n+1}\|_V \le k^{1/2} K_8(\mathcal{N}_3)e^{K_9(\alpha_1+k_0)} ,$$

where $K_8(\mathcal{N}_3)$ and K_9 are positive constants.

Hence, we have:

$$(5.34)\qquad \max_{\alpha_0/k\le n\le m+1} \|T(nk)u^0 - T_n^k u^0\|_V \le \frac{k^{1/2}}{\alpha_0} K_8(\mathcal{N}_3)e^{K_9(\alpha_1+k_0)}$$

And Theorem 5.1 is proven.

Remark 5.2: If f is globally Lischitz continuous from H into H, one can improve the estimate (5.34) (see [Crouzeix and Thomée (1)]).

Remark 5.3: Now let us consider a discretization in space and time of the equation (5.1). More precisely, if $(V_h)_h$ are the spaces given in Section 4.1, we define an approximation $u_n^h \in V_h$ of the solution u of (5.1) at the time t_n by the recursion formula

$$(5.1)_k^h \qquad \begin{cases} u_{n+1}^h + (1 - (1-\theta)kA_h)(1 + \theta kA_h)^{-1} u_n^h + k(1 + \theta kA_h)^{-1}Q_h f(u_n^h)) \\ \\ u_{oh} = u_h^0 \in V_h, \end{cases}$$

(where A_h and Q_h are given in Section 4.1).

Then in the same way as above, one proves that $(5.1)_k^h$ gives rise to a dynamical system T_k^h which admits an attractor $\mathcal{A}_k^h$. And $\delta_V(\mathcal{A}_k^h,\mathcal{A}) \to 0$ as h and k tend to 0.

Furthermore, if we are in the situation described in Section 4.2 and if $kh^{-2m} \leq C$ where C is a positive constant, one can define a dynamical system $\tilde{T}_k^h$ which admits an attractor $\tilde{\mathcal{A}}_k^h$ in Y^α and $\delta_{Y^\alpha}(\tilde{\mathcal{A}}_k^h,t) \to 0$ as h and k tend to 0.

6. A Remark on the Two-Dimensional Navier-Stokes Equations.

Let Ω be a regular, bounded domain in $\mathbf{R}^2$. The Navier-Stokes equations for the velocity $u(x,t) = (u_1(x,t),u_2(x,t))$ and the pressure $p(x,t)$, are

$$
(6.1)\quad \begin{cases}
\dfrac{\partial u}{\partial t} - \nu\Delta u + \displaystyle\sum_{i=1}^{2} u_i \frac{\partial u}{\partial x_i} + \operatorname{grad} p = F & \text{in } \Omega \times \mathbf{R}_+, \\
\operatorname{div} u = 0 & \text{in } \Omega \times \mathbf{R}_+ \\
u = 0 & \text{on } \partial\Omega \times \mathbf{R}_+ \\
u(x,0) = u_0(x) & \text{in } \Omega,
\end{cases}
$$

where F and u_0 are given and $\nu > 0$ is the kinematic viscosity. Let us denote by $\mathbf{H}^j(\Omega)$ the space $(H^j(\Omega))^2$ for $j = 1$ or 2 and by $\mathbf{L}^2(\Omega)$ the space $(L^2(\Omega))^2$. We consider the space

$$U = \{\phi \in (C_0^\infty(\Omega))^2;\ \operatorname{div} \phi = 0\}$$

and denote by H and V the closures of U in $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1(\Omega)$ respectively. The spaces H and V are provided with the inner products

$$(u,v) = \sum_{j=1}^{2} \int_\Omega u_j v_j dx\ ,$$

and

$$((u,v)) = \sum_{j,k=1}^{2} \int_\Omega \frac{\partial u_k}{\partial x_j} \frac{\partial v_k}{\partial x_j}\, dx$$

respectively, where $x = (x_1,x_2)$.

We also set $|u| = (u,u)^{1/2}$ and $\|u\| = ((u,u))^{1/2}$ for u in H and V respectively.

Let us denote by P the orthogonal projection of $\mathbf{L}^2(\Omega)$ onto H. We define $A = -P\Delta$ to be the operator with domain $D(A) = \mathbf{H}^2(\Omega) \cap V$ acting in H and use the same notation for its extension to an operator from V into V'. Since A^{-1} is a compact self-adjoint linear operator in H, the spectrum of A consists of an infinite sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \dots$$

of eigenvalues (counted according to their multiplicities) $\lambda_n \to \infty$ as $n \to \infty$ and there exists an orthonormal basis $\{\phi_n\}_{n \geq 1}$ of H such that

$$A\phi_n = \lambda_n \phi_n, \qquad n = 1,2,\dots$$

For any $N \geq 1$, we denote by P_N the orthogonal projection in H (and in V,V',D(A)) onto the space V_N spanned by $\phi_1, \phi_2, \dots, \phi_N$. We recall that

$$(6.2) \qquad \begin{cases} \|v\| \geq \lambda_1^{1/2}|v|, & \forall v \in V, \\ \\ |Av| \geq \lambda_1^{1/2} \|v\|, & \forall v \in D(A). \end{cases}$$

As Ω is a regular domain, we also have:

$$(6.3) \qquad c\,\|u\|_{\mathbf{H}^2(\Omega)} \leq |Au| \leq c'\|u\|_{\mathbf{H}^2(\Omega)}, \quad \forall u \in D(A).$$

For $u = (u_1,u_2)$ and $v = (v_1,v_2)$ in $\mathbf{H}^1(\Omega)$ we define $B(u,v) \in V'$ by

$$(6.4) \qquad (B(u,v),w) = \sum_{j,k=1}^{2} \int_\Omega u_j \frac{\partial v_k}{\partial x_j} w_k \, dx, \quad \forall w \in V$$

Then B is a bilinear continuous operator from $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ into V' and this operator can be extended as an operator from $\mathbf{H}^{m_1}(\Omega) \times \mathbf{H}^{m_2}(\Omega)$ into V' or H, for appropriate values of m_1 and m_2 (see [Témam], for instance). Subsequently, we shall use in the following inequality

$$|(B(u,v),w)| \leq c_1|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}\,|Av|^{1/2}|w|, \quad \forall u \in V,\ v \in D(A),\ w \in H. \tag{6.5}$$

Finally let us recall that

$$(B(u,v),w) = -(B(u,w),v), \quad \forall u,v,w \in V.$$

Using the above notations it can be shown that (6.1) is equivalent to the following initial value problem

$$\begin{cases} \dfrac{du}{dt} + \nu\, Au + B(u,u) = f \quad \text{in} \quad H, \\ u(0) = u_0\,, \end{cases} \tag{6.6}$$

where we assume that $f(x) = PF(x)$ and u_0 belong to H and V respectively (see [Temam] for further details). Let us point out that we assume that f does not depend on t.

Now we introduce the map $T(t)\colon V \to V$, $t \geq 0$, defined by $T(t)u_0 = u(t)$ where u(t) is the solution of (6.6). It is well known that $T(t)u_0$ exists for any $t \geq 0$ and any $u_0 \in V$ and that T(t) is a C^0-semigroup on V (see [Ladyzhenskaya (1) and (2)], for instance). In the same papers, she also showed that $T(t)u_0$ has its $\overline{\lim}$ as $t \to +\infty$ bounded by a constant independent of the initial data, i.e., T(t) is point dissipative. Since T(t) is compact for $t > 0$, we deduce from a result of [Billotti and LaSalle] that T(t) admits a compact attractor $\mathcal{A}$ which attracts bounded sets of V (see [Hale, 2] also).

Now let us consider the following differential system on the space V_N spanned by $\phi_1, \phi_2, \ldots, \phi_N$:

$$\begin{cases} \dfrac{du_N}{dt} + \nu\, Au_N + P_N B(u_N,u_N) = P_N f(x) \\ u_N(0) = u_{0N}\,, \end{cases} \tag{6.6_N}$$

where $u_{0N} \in V_N$. We introduce the map $T_N(t) : V_N \to V_N$, $t \geq 0$, defined by $T_N(t)u_{0N} = u_N(t)$ where $u_N(t)$ is the solution of $(6.6)_N$. As above $T_N(t)$ is a C^0-semigroup on V_N (see [Témam] for instance). In [Témam, §14.2], it is also shown that $T_N(t)u_{0N}$ has its $\overline{\lim}$ as $t \to +\infty$ bounded by a constant independent of the initial data and of N. Thus, by [Billotti and LaSalle] $T_N(t)$ admits a compact attractor A_N which attracts bounded sets of V_N. But thanks to Theorem 2.4 we obtain a more precise result given in Theorem 6.1. For related results, see [Constantin, Foias, Témam].

Theorem 6.1. *For any* $N \geq 1$, T_N *admits a compact attractor* A_N *which attracts bounded sets of* V_N. *Moreover,* $\delta_X(A_N,A) \to 0$ *as* $N \to +\infty$.

Proof. We are going to show that the hypotheses of Lemma 2.1 are satisfied. Let $t_0 > 0$ be a real number and N_1 be a bounded open neighborhood of t. We shall prove that $T_N(t)$ approximates $T(t)$ on N_1 uniformly on compact sets of $[A_0,+\infty)$.

We set : $B_0 = \max_{v \in N_1} \| v \|$. By [Temam, lemma 11.1 and lemma 14.3], we have, for any $N \geq 1$, for any $u_{0N} \in N_1 \cap V_N$,

$$(6.7) \qquad \sup_{t \geq 0} (\sup(\|T(t)u_{0N}\|, \|T_N(t)u_{0N}\|) \leq C_0,$$

and

$$(6.8) \qquad \sup_{t \geq t_0} |AT(t)u_{0N}| \leq K_0,$$

where $C_0 \equiv C_0(B_0)$ is a positive constant depending on B_0 only and $K_0 \equiv K_0(B_0,t_0)$ is a positive constant depending on B_0 and t_0 only.

Now we set $u(t) = T(t)u_{0N}$ and $u_N(t) = T_N(t)u_{0N}$. Let $t_1 > t_0$ be a real number; we at first estimate $\|u_N(t) - P_N u(t)\|$ for $0 \leq t \leq t_1$. The function $u_N - P_N u$ satisfies:

$$(6.9) \qquad \frac{d}{dt}(u_N - P_N u) + \nu A(u_N - P_N u) = P_N B(u,u) - P_N B(u_N,u_N)$$

Taking the inner product in H of (6.9) by $A(u_N - P_N u)$, we obtain:

$$(6.10)\qquad \frac{1}{2}\frac{d}{dt}\,||u_N - P_N u||^2 + \nu|A(u_N - P_N u)|^2$$

$$= (B(u,u-P_N u) + B(u,P_N u-u_N) + B(u-P_N u,u_N)$$

$$+ B(P_N u-u_N,u_N),\ A(u_N-P_N u)).$$

Thanks to the estimate (6.5), (6.10) we obtain

$$(6.11)\qquad \frac{1}{2}\frac{d}{dt}\,||u_N - P_N u||^2 + \nu|A(u_N - P_N u)|^2$$

$$\leq c_1|u|^{1/2}||u||^{1/2}||u - P_N u||^{1/2}|A(u - P_N u)|^{1/2}|A(P_N u - u_N)|$$

$$+ c_1|u|^{1/2}||u||^{1/2}||u_N - P_N u||^{1/2}|A(u_N - P_N u)|^{3/2}$$

$$+ c_1|u - P_N u|^{1/2}||u - P_N u||^{1/2}||u_N||^{1/2}|Au_N|^{1/2}|A(u_N - P_N u)|$$

$$+ c_1|P_N u - u_N|^{1/2}\,||P_N u - u_N||^{1/2}||u_N||^{1/2}|\,Au_N|^{1/2}|A(u_N - P_N u)|.$$

Using Young's inequality in the form

$$ab \leq \epsilon a^p + c_\epsilon b^{p'},\ \ 1 < p < +\infty,\ \ \epsilon > 0,\ \ p' = \frac{p}{p-1}\ ,\ \ c_\epsilon = \frac{p-1}{p^{p'}\epsilon^{1/p-1}}$$

with $p = \frac{4}{3}$ and $\epsilon = \frac{\nu}{4}$ and with $p = 2$ and $\epsilon = \frac{\nu}{4}$, we infer from (6.11),

$$\frac{d}{dt}\,||u_N - P_N u||^2 \leq c_2[\ |u|\ ||u||\ ||u - P_N u||\ |A(u - P_N u)|$$

$$+ |u|^2||u||^2\ ||u_N - P_N u||^2$$

$$+ ||u_N||\ |Au_N|\ |u - P_N u|\ ||u - P_N u||$$

$$+ ||u_N||\ |Au_N|\ ||P_N u - u_N||^2],$$

or also, by (6.7),

$$(6.12) \qquad \frac{d}{dt}||u_N - P_N u||^2 \leq c_3||u_N - P_N u||^2[\,1 + |Au_N|\,]$$

$$+ c_3\,||u - P_N u||[\,|A(u - P_N u)| + |Au_N|\,|u - P_N u|\,]$$

where $c_3 \equiv c_3(c_0)$ is a positive constant depending on c_0 only. But (6.12) is a differential inequality of the form

$$z' \leq a + bz$$

By Gronwall's lemma, this inequality yields

$$z(t) \leq z(0)\,e^{\int_0^t b(T)dT} + \int_0^t a(s)e^{\int_s^t b(T)dT}\,ds,$$

which gives, in our case, for $0 \leq t \leq t_1$,

$$(6.13) \qquad ||u_N(t) - P_n u(t)||^2 \leq e^{c_3\int_0^{t_1}(1+|Au_N|)ds} \times$$

$$\times \int_0^{t_1}\{|u-P_N u|\,||u-P_N u||\,|Au_N| + ||u-P_N u||\,|A(u-P_N u)|\}ds$$

Using the properties of P_N and the Cauchy-Schwarz inequality, we deduce from (6.13) that

$$(6.14) \qquad ||u_N(t) - P_N u(t)||^2 \leq c_3\,e^{c_3 t_1(1+(\int_0^{t_1}|Au_N|^2 ds)^{1/2})} \times$$

$$\times \left(\int_0^{t_1}||u - P_N u||^2 ds\right)^{1/2}\left[2C_0\left(\int_0^{t_1}|Au_N|^2\right)^{1/2} + 2\left(\int_0^{t}|Au|^2 ds\right)^{1/2}\right].$$

Arguing as in [Temam, §3.1], one proves that, for $N \geq 1$,

$$(6.15) \qquad \sup\left\{\int_0^{t_1}|Au_N|^2 dt,\ \int_0^{t_1}|Au|^2 dt\right\} \leq c_4(t_1,c_0)$$

where $c_4(t_1,c_0)$ is a positive constant depending on t_1 and c_0 only. Thanks to (6.15), we can write:

$$\int_0^{t_1} ||u - P_N u||^2 ds \leq \frac{c_4(t_1,c_0)}{\lambda_{N+1}} . \tag{6.16}$$

Finally from (6.14), (6.15) and (6.16) we derive, for $0 \leq t \leq t_1$,

$$||u_N(t) - P_N u(t)||^2 \leq \frac{c_5(t_1, c_0)}{\lambda_{N+1}} , \tag{6.17}$$

where $c_5(t_1,c_0)$ is a positive constant depending on t_1 and c_0 only.

Now we want to estimate $||u(t) - P_N u(t)||^2$ for $t_0 \leq t \leq t_1$. Using (6.8), we prove that, for $t_0 \leq t \leq t_1$,

$$||u(t) - P_N u(t)||^2 \leq \frac{k_0^2}{\lambda_{N+1}} . \tag{6.18}$$

Finally, as $\lambda_{N+1} \to +\infty$ as $N \to +\infty$, we deduce from (6.17) and (6.18) that $T_N(t)$ approximates $T(t)$ on $\mathcal{N}_1$ uniformly on compact sets of $[t_0,+\infty)$.

7. Approximation of the Damped Wave Equation.

Let Ω be a bounded domain in $\mathbf{R}^3$, α be a positive constant and consider the equation

$$(7.1)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} + 2\alpha \dfrac{\partial u}{\partial t} - \Delta u = -f(u) - g(x) & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega, \\ \left(u, \dfrac{\partial u}{\partial t}\right)_{/t=0} = (\phi,\psi), & \end{cases}$$

where g belongs to $L^2(\Omega)$ and (ϕ,ψ) belongs to the space $X \equiv H_0^1(\Omega) \times L^2(\Omega)$. We assume that the boundary $\partial\Omega$ of Ω is smooth enough or that Ω is a convex domain. Furthermore, we suppose that $f \in C^3(\mathbf{R})$ and that there are constants $\gamma > 0$, $c_1 > 0$ such that,

$$(7.2)\qquad \begin{cases} |f(v)| \leqslant C_1(|v|^{3-\gamma} + 1),\ |f'(v)| \leqslant C_1(|v|^{2-\gamma} + 1), \\ |f''(v)| \leqslant C_1(|v| + 1),\ |f'''(v)| \leqslant C_1, \text{ for all } v \in \mathbf{R}. \end{cases}$$

Inequalities (7.2) imply that the map f: $\phi \in H_0^1(\Omega) \to f(\phi(x)) \in L^2(\Omega)$ is a compact, C^2-mapping from $H_0^1(\Omega)$ into $L^2(\Omega)$. Henceforth, we equip the space X with the norm

$$(7.3)\qquad \|(\phi,\psi)\|_X = (\|\phi\|^2_{H_0^1(\Omega)} + \|\psi\|^2_{L^2(\Omega)})^{1/2}, \quad \forall(\phi,\psi) \in X$$

As it was proven in [Babin and Vishik], for any $(\phi,\psi) \in X$, Problem (7.1) has a unique solution u(t), for $t \geqslant 0$, and the pair $(u,\frac{\partial u}{\partial t})$ belongs to $C^0([0,+\infty);X)$. Furthermore, if we set $T(t)(\phi,\psi) = (u(t),\frac{\partial u}{\partial t}(t))$, for $t \geqslant 0$, then $T(t) : X \to X$, $t \geqslant 0$, is a C^0-semigroup on X.

Now suppose there is a constant $c > 0$ so that f satisfies

(7.4) $\qquad f(v)v > -c,\ f'(v) > -c, \quad \text{for all } v \in \mathbf{R}.$

Let us introduce the Liapunov functional V given by

$$V(\phi,\psi) = \int_\Omega \left[\frac{1}{2}\,|\nabla\phi(x)|^2 + \frac{1}{2}\psi^2(x) + F(\phi(x)) + g(x)\phi(x)\right] dx,$$

for all $(\phi,\psi) \in X$, where $F(v) = \int_0^v f(s)ds$. It was proven in [Babin and Vishik] that

$$\text{(7.5)} \qquad \begin{cases} V(\phi,\psi) \geq \frac{1}{2}\|\psi\|^2_{L^2(\Omega)} + \|\phi\|^2_{H_0^1(\Omega)} - C_2, \\[2ex] V(\phi,\psi) \leq \frac{1}{2}\|\psi\|^2_{L^2(\Omega)} + C_3\|\phi\|^4_{H_0^1(\Omega)} + C_4, \end{cases}$$

where C_2, C_3, C_4 are some fixed positive constants, and that, for $t \geq \tau$ and for any solution u of (7.1)

$$\text{(7.6)} \qquad V(u(t,\cdot),\frac{\partial u}{\partial t}(t,\cdot)) - V(u(\tau,\cdot),\frac{\partial u}{\partial t}(\tau,\cdot)) = -2\alpha \int_\tau^t \int_\Omega \left[\frac{\partial u(s,x)}{\partial t}\right]^2 dx\, ds$$

The estimates (7.5) imply that the orbits of bounded sets are bounded. In particular, there exist two functions $C_0(R)$ and $C_1(R)$ of R such that, if

(7.7) $\qquad \|(\phi,\psi)\|_X^2 \leq R^2,$

then,

(7.8) $\qquad V(T(t)(\phi,\psi)) \leq V(\phi,\psi) \leq C_0(R), \quad \forall t \in \mathbf{R}$

and

(7.9) $\qquad \|T(t)(\phi,\psi)\|_X \leq C_1(R), \qquad\qquad \forall t \in \mathbf{R}.$

Moreover, it was shown in [Hale, 2] that T(t) is point dissipative and is an α-contraction. Therefore, due to a result of [Massat], T(t) admits a compact attractor A in X, which attracts bounded sets of X (see [Hale (2), Theorem 6.1]).

7.1 Approximation by a special projection method.

Let us recall that the spectrum of the operator $-\Delta$ with domain $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ consists of an infinite sequence

$$0 < \lambda_1 \leq \lambda_2 \leq ..$$

of eigenvalues (counted according to their multiplicities), $\lambda_n \to +\infty$ as $n \to +\infty$ and that there exists an orthonormal basis $\{w_n\}_{n\geq 1}$ of $L^2(\Omega)$ such that

$$(7.10) \qquad -\Delta w_n = \lambda_n w_n .$$

Note that $\{\lambda_n^{-1/2} w_n\}_{n\geq 1}$ is an orthonormal basis of $H_0^1(\Omega)$. For any $N \geq 1$, we denote by P_N the *orthogonal projection* in $L^2(\Omega)$ (and in $H_0^1(\Omega)$) *onto* the space V_N spanned by $w_1, w_2, ..., w_N$, and we consider the following equation in V_N:

$$(7.1)_N \quad \begin{cases} \dfrac{\partial^2 u_N}{\partial t^2} + 2\alpha \dfrac{\partial u_N}{\partial t} - \Delta u_N = -P_N f(u_N) - P_N g(x), \\ \\ \left[u_N, \dfrac{\partial u_N}{\partial t}\right]_{/t=0} = (\phi_N, \psi_N), \end{cases}$$

where (ϕ_N,ψ_N) belongs to the space $X_N = V_N \times V_N$. We can prove, as for the problem (7.1), that, for any (ϕ_N, ψ_N) in X_N, the equation $(7.1)_N$ has a unique solution $u_N(t)$, for $t \geq 0$. Moreover, if we set $T_N(t)(\phi_N,\psi_N) = (u_N(t), \partial u_N(t)/\partial t$, for $t \geq 0$, then $T_N(t) : X_N \to X_N$, $t \geq 0$, is a C^0-semigroup on X_N.

Theorem 7.1. For any $N \geq 1$, T_N admits a compact attractor A_N which attracts bounded sets of X_N. Moreover, $\delta_X(A_N, A) \to 0$ as $N \to +\infty$.

Proof.

1) We at once verify that, for $t \geq \tau$, for any solution u_N of Equation $(7.1)_N$,

$$(7.11)\qquad V(u_N(t,\cdot), \frac{\partial u_N}{\partial t}(t,\cdot)) - V(u_N(\tau,\cdot), \frac{\partial u_N}{\partial t}(\tau,\cdot)) = -2\alpha \int_\tau^t \int_\Omega \frac{\partial u_N}{\partial t}(s,x)ds.$$

The estimates (7.5) imply that the orbits of bounded sets are bounded independently of N. In particular, $T_N(t)(\phi_N,\psi_N)$ satisfies the estimates (7.8) and (7.9), for any (ϕ_N,ψ_N) satisfying (7.7). As $T_N(t)$ is compact, the orbit through (ϕ_N,ψ_N) is precompact and its ω-limit set must be an invariant set. Relation (7.11) implies that its ω-limit set belongs to the set E_N of the equilibrium points. Using the condition (7.4), one easily proves that there exists a constant $r_0 > 0$ such that

$$(7.12)\qquad \forall N \geq 1, \qquad E_N \subset B_X(r_0),$$

where, for any $r > 0$, $B_X(r) = \{(\phi,\psi)\ \ X: \ \|(\phi,\psi)\|_X < r\}$. Let us also set $B_{X_N}(r) = B_X(r) \cap X_N$. Then, for $r_1 = 2r_0$, the ball $B_{X_N}(r_1)$ attracts all points of X_N (i.e., for any $(\phi_N,\psi_N) \in X_N$, there exists $t_N \geq 0$ such that, for $t \geq t_N$, $T_N(t)(\phi_N,\psi_N) \in B_{X_N}(r_1)$). Let us remark that the orbit of $B_{X_N}(r_1)$ is included in $B_{X_N}(C_1(r_1))$, where $C_1(r_1)$ is given by (7.9), and that $B_{X_N}(C_1(r_1))$ attracts a neighborhood of any point and, hence, all compact sets of X_N. We now set: $R_0 = C_1(C_1(r_1))$. Arguing as in [Hale, 1, Theorem 2.1], one finally shows that $T_N(t)$ admits a compact attractor $\mathcal{A}_N$ which attracts bounded sets of X_N and is included in the ball $B_X(R_0) \cap X_N$.

2) In order to prove that $\delta_X(\mathcal{A}_N,\mathcal{A}) \to 0$ as $N \to +\infty$, we show that the hypotheses of Lemma 2.1 hold. Let $\mathcal{N}_1 \equiv B_X(R_1)$ be a neighborhood of $\mathcal{A}$. We shall prove that $T_N(t)$ approximates $T(t)$ on $\mathcal{N}_1$ uniformly on compact sets of $[0,+\infty)$. Let t_1 be any real number. We at first estimate $\|(u(t) - P_N u(t), \frac{\partial u}{\partial t}(t) - \frac{\partial P_N u}{\partial t}(t))\|_X$ for $0 \leq t \leq$ where $(u(t),\frac{\partial u}{\partial t}(t)) = T(t)(\phi_N,\psi_N)$ and $(\phi_N,\psi_N) \in \mathcal{N}_1 \cap X_N$. We have:

$$(7.13)\qquad \frac{\partial^2}{\partial t^2}(u - P_N u) + 2\alpha \frac{\partial}{\partial t}(u - P_N u) - \Delta(u - P_N u) = -(I - P_N)f(u) - (I - P_N)g(x).$$

Taking the inner product in $L^2(\Omega)$ of (7.13) by $\frac{\partial}{\partial t}(u - P_N u)$, we get after an integration from 0 to t:

$$(7.14)\qquad \left\|\frac{\partial}{\partial t}(u - P_N u)(t)\right\|^2_{L^2(\Omega)} + \|u(t) - P_N u(t)\|^2_{H_0^1(\Omega)}$$

$$\leq \frac{t_1}{\alpha}\Big(\sup_{s\in[0,t_1]} \|(I-P_N)f(u(s))\|^2_{L^2(\Omega)} + \|(I-P_N)g(x)\|^2_{L^2(\Omega)}\Big)$$

Since f is a compact mapping from $H_0^1(\Omega)$ into $L^2(\Omega)$ and u(s), $0 \leq s \leq t$, belongs to the bounded set $\bar{B}(C_1(R_1)) = \{v \in H_0^1(\Omega) : \|v\|_{H_0^1(\Omega)} \leq C_1(R_1)\}$, we deduce from (7.14) that, for $0 \leq t \leq t_1$,

$$(7.15)\qquad \left\|\frac{\partial}{\partial t}(u - P_N u)(t)\right\|^2_{L^2(\Omega)} + \|u(t) - P_N u(t)\|^2_{H_0^1(\Omega)} \leq \eta_1(N,t_1,C_1(R_1))$$

where

$$(7.16)\qquad \lim_{N\to+\infty} \eta_1(N,t_1,C_1(R_1)) = 0.$$

Now we estimate $\|(P_N u(t) - u_N(t), \frac{\partial}{\partial t}(P_N u(t) - u_N(t)))\|_X$ for $0 \leq t \leq t_1$, where $(u_N(t), \frac{\partial u_N}{\partial t}(t)) = T_N(t)(\phi_N,\psi_N)$. The function $u_N - P_N u$ satisfies the equation

$$(7.17)\qquad \frac{\partial^2}{\partial t^2}(u_N - P_N u) + 2\alpha\,\frac{\partial}{\partial t}(u_N - P_N u) - \Delta(u_N - P_N u) = P_N(f(u) - f(u_N))$$

Taking the inner product in $L^2(\Omega)$ of (7.17) with $\frac{\partial}{\partial t}(u_N - P_N u)$, we obtain

$$(7.18)\qquad \frac{1}{2}\frac{\partial}{\partial t}\left\|\frac{\partial}{\partial t}(u_N - P_N u)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\left(\|u_N - P_N u\|^2_{H_0^1(\Omega)}\right)$$

$$\leq \frac{L^2}{2\alpha}\|u_N - P_N u\|^2_{H_0^1(\Omega)} + \frac{L^2}{2\alpha}\|u - P_N u\|^2_{H_0^1(\Omega)},$$

where $L > 0$ is the Lipschitz constant of f in the ball $\bar{B}(C_1(R_1))$. Now using Gronwall's lemma, we derive from (7.18) as well as from (7.15) that, for $0 \leq s \leq t$,

$$(7.20)\qquad \left\|\frac{\partial}{\partial t}(u_N - P_N u)(t)\right\|^2_{L^2(\Omega)} + \|u_N(t) - P_N u(t)\|^2_{H_0^1(\Omega)} \leq e^{\frac{t_1 L^2}{2\alpha}}\,\frac{L^2}{2\alpha}\,\eta_1(N,t_1,C_1(R_1))$$

The estimates (7.15), (7.16) and (7.20) show that $T_N(t)$ approximate $T(t)$ on $\mathcal{N}_I$ uniformly on compact sets of $[0,+\infty)$.

7.2. <u>A more general Galerkin method</u>.

Let $h > 0$ be a real parameter which will tend to 0 and $(V_h)_h$ be a family of finite-dimensional subspaces of $H_0^1(\Omega)$. We denote by $[\cdot,\cdot]$ the inner product of $L^2(\Omega)$ and by $a(\cdot,\cdot)$ the inner product of $H_0^1(\Omega)$, i.e.,

$$\forall v \in H_0^1(\Omega),\ \forall w \in H_0^1(\Omega),\ a(v,w) = \int_\Omega \nabla v\ \nabla w\ dx.$$

As in Section 4.1, we denote by $Q_h \in \mathcal{L}(L^2(\Omega);V_h)$ and $P_h \in \mathcal{L}(H_0^1(\Omega);V_h)$ the orthogonal projectors on V_h in the spaces $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively. We also introduce the operator $A_h \in \mathcal{L}(V_h;V_h)$ defined by

$$\forall v_h \in V_h,\ (A_h w_h, v_h) = a(w_h,v_h) \quad \text{for} \quad w_h \in V_h.$$

We consider the following equation in V_h

$$(7.1)_h \qquad \begin{cases} \dfrac{\partial^2 u_h}{\partial t^2} + 2\alpha\ \dfrac{\partial u_h}{\partial t} + A_h\ u_h = -Q_h f(u_h) - Q_h g(x), \\[2ex] \left(u_h, \dfrac{\partial u_h}{\partial t}\right)_{/t=0} = (\phi_h,\psi_h)\ , \end{cases}$$

where (ϕ_h,ψ_h) belongs to the space $X_h = V_h \times V_h$. As in Section 7.1, we introduce the map $T_h(t)\colon X_h \to X_h$, for $t \geq 0$, defined by $T_h(\phi_h,\psi_h) = (u_h(t),\frac{\partial u_h}{\partial t}(t))$ where u_h is the solution of $(7.1)_h$. So we obtain a C^0-semigroup on X_h. As in Section 4.1, we need some additional hypotheses on the spaces $(V_h)_h$:

(7.21)(i) there exists a constant $K_0 > 0$, independent of h, such that, for any $h > 0$,

$$||Q_h||_{\mathcal{L}(H_0^1(\Omega);H_0^1(\Omega))} \leq K_0$$

and

(7.21)(ii) there exist two constants $K_A > 0$ and $\theta > 0$, independent of h, such that, for any w in $H_0^1(\Omega)$,

$$||w - P_h w||_{L^2(\Omega)} + ||w - Q_h w||_{L^2(\Omega)} \leq K_1 h^{\theta} ||w||_{H_0^1(\Omega)}.$$

Finally we introduce the Hilbert space $Y \equiv L^2(\Omega) \times H^{-1}(\Omega)$, normed by $||(\phi,\psi)||_Y = (||\phi||^2_{L^2} + ||\psi||^2_{H^{-1}(\Omega)})^{1/2}$.

Now we are able to prove the following result.

Theorem 7.2. *For any* $h > 0$, T_h *admits a compact attractor* $\mathcal{A}_h$ *which attracts bounded sets of* X_h *and is contained in the ball* $B_X(R_0) \cap X_h$, *where* R_0 *is a constant independent of* h. *Moreover,* $\delta_Y(\mathcal{A}_h,\mathcal{A}) \to 0$ *as* $h \to 0$.

Remark 7.3. In Section 4, we proved that $\delta_V(\mathcal{A}_h,\mathcal{A}) \to 0$ as $h \to 0$. Here, we can no longer prove that $\delta_X(\mathcal{A}_h,\mathcal{A}) \to 0$ as $h \to 0$, because T(t) has no longer a smoothing action.

Proof of Theorem 7.2.

1) At first we show in the same way as in the proof of Theorem 7.1 that, for any $h > 0$, T_h admits a comapct attractor $\mathcal{A}_h$ which attracts bounded sets of X_h and is contained in $B_X(R_0) \cap X_h$, where R_0 is a constant independent of h. Remark that R_0 can be chosen so that $\mathcal{A}$ is also contained in $B_X(R_0)$.

2) Now let us check that, for any $r > 0$, there exists a constant $L(r) > 0$ such that, for all v and w in the ball $\bar{B}(r) = \{v \in H_0^1(r) : ||v||_{H_0^1(\Omega)} \leq r\}$, we have

(7.22) $$||f(v) - f(w)||_{H^{-1}(\Omega)} \leq L(r)||v - w||_{L^2(\Omega)}.$$

Indeed we can write

$$||f(v) - f(w)||_{H^{-1}(\Omega)} = \sup_{\Phi\in H_0^1(\Omega)} \frac{\int_\Omega (f(v(x)) - f(w(x)))\Phi(x)dx}{||\Phi||_{H_0^1(\Omega)}}$$

$$\leq \sup_{\Phi\in H_0^1(\Omega)} \frac{\int_\Omega\int_0^1 f'(w(x)+\tau(v(x)-w(x)))(v(x)-w(x))\Phi(x)dxds}{||\Phi||_{H_0^1(\Omega)}}$$

Hence, using the hypothesis (7.2), we obtain

$$(7.23) \qquad ||f(v) - f(w)||_{H^{-1}(\Omega)} \leq \sup_{\Phi\in H_0^1(\Omega)} \frac{c_1}{||\Phi||_{H_0^1(\Omega)}} \left\{\left[\int_\Omega |v(x) - w(x)|^2dx\right]^{1/2} \times \right.$$

$$\left. \left[\int_\Omega 2(|v(x)|^\beta + |w(x)|^\beta + 1)dx\right] \left[\int_\Omega |\Phi(x)|^6dx\right]^{1/6}\right\},$$

where $\beta = \sup(3,6-3\gamma)$. As $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, the property (7.22) is a direct consequence of (7.23).

3) Now, for any $t_1 > 0$, we estimate $||(u(t) - u_h(t),\frac{\partial u}{\partial t}(t) - \frac{\partial u_h}{\partial t}(t))||_Y$ for $0 \leq t \leq t_1$, where $u(t)$ and $u_h(t)$ are the solutions of the equations (7.1) and $(7.1)_h$, respectively, with initial condition $(\phi_h,\psi_h) \in B_X(R_0)$. Thanks to the hypothesis (7.21)(ii), we have, on the one hand,

$$(7.24) \qquad ||u(t) - Q_hu(t)||_{L^2(\Omega)} \leq K_1h^\theta C_1(R_0),$$

and, on the other hand,

$$||\frac{\partial u}{\partial t}(t) - \frac{\partial}{\partial t} Q_hu(t)||_{H^{-1}(\Omega)} = \sup_{v\in H_0^1(\Omega)} \frac{\left[\frac{\partial u}{\partial t} - \frac{\partial}{\partial t} Q_hu,\ v - Q_h v\right]}{||v||_{H_0^1(\Omega)}}$$

$$\leq \sup_{v\in H_0^1(\Omega)} ||\frac{\partial u}{\partial t} - \frac{\partial}{\partial t} Q_hu||_{L^2(\Omega)}||v - Q_hv||_{L^2(\Omega)}$$

which gives

$$(7.25) \qquad \left|\left|\frac{\partial u}{\partial t}(t) - \frac{\partial}{\partial t} Q_h u(t)\right|\right|_{H^{-1}(\Omega)} \leq 2K_1 h^{\theta} C_1(R_0).$$

It remains to estimate the term $\|(Q_h u(t) - u_h(t), \frac{\partial}{\partial t} Q_h u(t) - \frac{\partial u_h}{\partial t}(t))\|_Y$ for $0 \leq t \leq t_1$. Note that the operator Q_h can be extended to a continuous, linear operator from $H^{-1}(\Omega)$ into V_h and that thus the element $u_h - Q_h u$ satisfies the equation:

$$(7.26) \qquad \frac{\partial^2}{\partial t^2} (u_h - Q_h u) + 2\alpha \frac{\partial}{\partial t} (u_h - Q_h u) + A_h(u_h - Q_h u) = -Q_h(f(u_h) - f(u)) - (A_h Q_h - Q_h A)u.$$

Let us introduce the operator $S_h \in \mathcal{L}(H^{-1}(\Omega);V_h)$ given by

$$(7.27) \qquad \forall f \in H^{-1}(\Omega),\ a(S_h f, v_h) = [f, v_h],\quad \forall v_h \in V_h.$$

Clearly, one has

$$(7.28) \qquad ||S_h f||_{H_0^1(\Omega)} \leq c||f||_{H^{-1}(\Omega)},$$

where $c > 0$ is a constant independent of h.

Taking the inner product in $L^2(\Omega)$ of (7.26) by $S_h(\frac{\partial}{\partial t}(u_h - Q_h u))$ and using the relation (7.27), we obtain:

$$(7.29) \qquad a(\frac{\partial^2}{\partial t^2} S_h(u_h - Q_h u), \frac{\partial}{\partial t} S_h(u_h - Q_h u)) + 2\alpha a(S_h \frac{\partial}{\partial t}(u_h - Q_h u), S_h \frac{\partial}{\partial t}(u_h - Q_h u)$$
$$+ a(u_h - Q_h u, S_h \frac{\partial}{\partial t}(u_h - Q_h u))$$
$$= -[f(u_h) - f(u), S_h \frac{\partial}{\partial t}(u_h - Q_h u)] + a(u - Q_h u, S_h \frac{\partial}{\partial t}(u_h - Q_h u))$$

But $a(u - Q_h u, S_h \frac{\partial}{\partial t}(u_h - Q_h u)) = a(P_h u - Q_h u, S_h \frac{\partial}{\partial t}(u_h - Q_h u))$

$$= [P_h u - Q_h u, \frac{\partial}{\partial t}(u_h - Q_h u)] = [P_h u - u, \frac{\partial}{\partial t}(u_h - Q_h u)]$$

and

$$a(u_h - Q_h u, S_h \frac{\partial}{\partial t}(u_h - Q_h u)) = [u_h - Q_h u, \frac{\partial}{\partial t}(u_h - Q_h u)].$$

Then, from (7.29) we can derive the following inequality:

$$\frac{1}{2}\frac{\partial}{\partial t}\left|\left|\frac{\partial}{\partial t} S_h(u_h - Q_h u))\right|\right|^2_{H^1_0(\Omega)} + 2\alpha\left|\left|\frac{\partial}{\partial t} S_h(u_h - Q_h u)\right|\right|^2_{H^1_0(\Omega)}$$

$$+ \frac{1}{2}\frac{\partial}{\partial t}\left|\left|u_h - Q_h u\right|\right|^2_{L^2(\Omega)}$$

$$\leq ||f(u) - f(u_h)||_{H^{-1}(\Omega)}\left|\left|\frac{\partial}{\partial t} S_h(u_h - Q_h u)\right|\right|_{H^1_0(\Omega)}$$

$$+ ||u - P_h u||_{L^2(\Omega)}\left|\left|\frac{\partial}{\partial t}(u_h - Q_h u)\right|\right|_{L^2(\Omega)}$$

Using the property (7.22) and the fact that $(u, \frac{\partial u}{\partial t})$ and $(u_h, \frac{\partial u_h}{\partial t})$ belong to $B_X(C_1(R_0))$, we infer from the above estimate that

$$\frac{\partial}{\partial t}\left|\left|\frac{\partial}{\partial t} S_h(u_h - Q_h u)\right|\right|^2_{H^1_0(\Omega)} + \frac{\partial}{\partial t}\left|\left|u_h - Q_h u\right|\right|^2_{L^2(\Omega)} \tag{7.30}$$

$$\leq \frac{L^2(C_1(R_0))}{\alpha}\left\{||u - Q_h u||^2_{L^2(\Omega)} + ||u_h - Q_h u||^2_{L^2(\Omega)}\right\}$$

$$+ 2C_1(R_0)||u - P_h u||_{L^2(\Omega)}.$$

Integrating (7.30) from 0 to t and using Gronwall's lemma as well as the hypothesis (7.21)(ii), we get, for $0 \leq t \leq t_1$,

$$\left|\left|\frac{\partial}{\partial t} S_h(u_h - Q_h u)(t)\right|\right|^2_{H^1_0(\Omega)} + ||(u_h - Q_h u)(t)||^2_{L^2(\Omega)} \leq K_2 t_1 e^{K_3 t_1} h^\theta, \tag{7.31}$$

where $K_2 > 0$ and $K_3 > 0$ are two constants depending on R_0 only. Now let us remark that

$$\left|\left|\frac{\partial}{\partial t} S_h(u_h - Q_h u)\right|\right|_{H^{-1}(\Omega)} = \sup_{v\in H_0^1(\Omega)} \frac{\left[\frac{\partial}{\partial t}(u_h - Q_h u), v\right]}{||v||_{H_0^1(\Omega)}}$$

$$= \sup_{v\in H_0^1(\Omega)} \frac{\left[\frac{\partial}{\partial t}(u_h - Q_h u), Q_h v\right]}{||v||_{H_0^1(\Omega)}}$$

$$= \sup_{v\in H_0^1(\Omega)} \frac{a(\frac{\partial}{\partial t} S_h(u_h - Q_h u), Q_h v)}{||v||_{H_0^1(\Omega)}}$$

and therefore, thanks to the hypothesis (7.21)(i),

$$(7.32)\qquad \left|\left|\frac{\partial}{\partial t} S_h(u_h - Q_h u)\right|\right|_{H^{-1}(\Omega)} \leq K_0 \left|\left|\frac{\partial}{\partial t} S_h(u_h - Q_h u)\right|\right|_{H_0^1(\Omega)}$$

Finally, by (7.24), (7.25), (7.31) and (7.32), we obtain, for $0 \leq t \leq t_1$,

$$(7.33)\qquad \left|\left|(u(t) - u_h(t), \frac{\partial u}{\partial t}(t) - \frac{\partial u_h}{\partial t}(t))\right|\right|_Y \leq K_4\, t_1^{1/2}\, e^{K_5 t_1}\, h^{\theta/2}\ ,$$

where K_4 and K_5 are positive constants depending on R_0 only.

4) Since, for any $h > 0$, $A_h \subset B_X(R_0)$, we deduce from the property (7.33), by arguing as in the proof of Proposiiton 2.10 (or in Remark 2.7), that, for any $\epsilon_0 >$ 0, there exists $h_0 > 0$ such that, for $h \leq h_0$, $\delta_Y(A_h, A) \leq \epsilon_0$.

Remark 7.1. The results of theorems 7.1 and 7.2 extend easily to the cases where Ω is a bounded domain in R or R^2 (for the conditions on f, see [Babin and Vishik] or [Hale, 2]).

References

Babin, A.V. and Vishik, M.I., Regular attractors of semigroups and evolution equations, J. Math. Pures et Appl. 62(1983), 441-491.

Billoti, J.E. and LaSalle, J.P., Periodic dissipative processes, Bull. Am. Math. Soc. 6(1971), 1082-1089.

Brenner, P., Crouzeix, M. and Thomée, V., Single step methods for inhomogeneous linear differential equations in Banach spaces, R.A.I.R.O. Anal. Numer. 16(1982), 5-26.

Ciarlet, P.G., *The Finite Element Method for Elliptic Problems*, North-Holland, 1978.

Constantin, P., Foias, C. and Témam, R., On the large time Galerkin approximation of the Navier-Stokes equations, SIAM J. Numer. Anal. 21(1984), 615-634.

Cooperman, G., α-Condensing Maps and Dissipative Systems, Ph.D. Thesis, Brown University, Providence, RI, June 1978.

Crouzeix, M. and Thomée, V. (1), On the discretization in time of semilinear parabolic equations with non-smooth initial data, Preprint of the University of Rennes (France), 1985.

Crouzeix, M. and Thomée, V. (2), private communication.

Fujita, H., and Mizutani, A., On the finite element method for parabolic equations I; approximation of holomorphic semigroups, J. Math. Soc. Japan 28(1976).

Hale, J.K. (1), Some recent results on dissipative processes, in Functional differential equations and bifurcation (Ize, ed.), Lecture Notes in Math., Vol. 799, Springer-Verlag (1980), 152-172.

Hale, J.K. (2), Asymptotic behavior and dynamics in infinite dimensions, in Research Notes in Math., Vol. 132, Pittman (1985), 1-42.

Hale, J.K., LaSalle, J.P. and Slemrod, M., Theory of a general class of dissipative processes, J. Math. Anal. Appl. 39(1972), 177-191.

Hale, J.K. and Lopes, O., Fixed point theorems and dissipative processes, J. Differential Equations 13(1973), 391-402.

Henry, D., Geometric theory of semilinear parabolic equations, Lecture Notes in Math., Vol. 840, Springer-Verlag, 1981.

Kato, T., Fractional powers of dissipative operators II, J. Math. Soc. Japan 14(1962), 242-248.

Khalsa, S.N.S., Finite element approximation of a reaction diffusion equation. Part I: Application of a topological technique to the analysis of asymptotic behavior of the semidiscrete approximation, preprint.

Ladyzhenskaya, O.A. (1), A dynamical system generated by the Navier-Stokes equations, Zapiski Nauk. Sem. Lenin. Gtd. Mat. Instituta Steklova 27(1972), 91-115.

Ladyzhenskaya, O.A. (2), Dynamical system generated by the Navier-Stokes equations, Soviet Physics Dokl. 17(1973), 647-649.

Lions, J.L., Espaces d'interpolation et domaines de puissances fractionnaires d'operateurs, J. Math. Soc. Japan 14(1962), 234-241.

Massat, P., Attractivity properties of α-contractions, J. Differential Equations 48(1983), 326-333.

Mora, X., Personal communication, Univ. Barcelona.

Raugel, G., Hilbert space estimates in the approximation of inhomogeneous parabolic problems by single step methods, submitted to Math. Comp.

Rutkowski, P., Approximate solutions of eigenvalue problems with reproducing nonlinearities, Z. Angew. Math. Phys. 34(1983), 310-321.

Schmitt, K., Thompson, R.C. and W. Walter, Existence of solutions of a nonlienar boundary value problem via the method of lines, Nonlinear Anal., TMA 2(1978), 519-535.

Témam, R., Navier-Stokes equations and nonlinear functional analysis, CBMS-NSF Vol. 41, SIAM (1983).

Thomée, V., Galerkin finite element methods for parabolic problems, Lecture Notes in Math., Vol. 1054, Springer-Verlag (1984).

Thomée, V., Wahlbin, L., On Galerkin methods in semilinear parabolic problems, SIAM J. Numer. Anal. 12(1975).

END
DTIC
8-86